The Mathematical Theory of Thin Film Evolution

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The Mathematical Theory of Thin Film Evolution

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DEDICATION

I dedicate this thesis to my wife, Semra.

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SUMMARY

We try to explain the mathematical theory of thin liquid film evolution. We start with introducing physical processes in which thin film evolution plays an important role. Derivation of the classical thin film equation and existing mathematical theory in the literature are also introduced.

To explain the thin film evolution we derive a new family of degenerate parabolic equations. We prove results on existence, uniqueness, long time behavior, regularity and support properties of solutions for this equation.

At the end of the thesis we consider the classical thin film Cauchy problem on the whole real line for which we use asymptotic equipartition to show $H^1(\mathbb{R})$ convergence of solutions to the unique self-similar solution.

CHAPTER I

INTRODUCTION

1.1 Background and Physical Processes

Thin-film type evolution processes arise in our everyday life ranging from very simple processes, such as rain drop movement on a window, to complicated industrial processes [19, 72]. Appearing in many physical processes used in industry, thin-film dynamics attracted researchers from physics, mathematics and various engineering departments recently. Because of the large number of applications in industry, understanding the evolution of a thin liquid film surface is extremely important. This may lead to a better design in these processes, making our lives easier. These equations are also an interesting source of math issues. Generally, these equations are fourth-order, degenerate, non linear, parabolic partial differential equations. The nature of these equations is very much different from their second order analogs in that there is no maximum principle for these higher order equations. Thus, one must rely on finding dissipated energies and entropies to prove rigorous mathematical results on these equations.

Before mentioning the physical processes, let us derive the most commonly used thin film equation. Let us consider a viscous capillary driven flow of thin droplet of thickness $h(t,x)$. Since the surface tension is the only driving force here, the evolution of the liquid surface is caused by the variation of the arclength. This means that the curvature causes to have the following formula for the pressure, which is the variation of the energy functional with respect to $h(t,x)$.

$$p(t,x) = -\frac{\partial^2 h(t,x)}{\partial x^2}.$$

Hence we have $\frac{\delta E_0}{\delta h} = p(t,x) = -\frac{\partial^2 h(t,x)}{\partial x^2}$, as the energy functional in this problem is $E_s = \int \sqrt{1+h_x^2}dx$ and this can be approximated by $E_0 = \frac{1}{2}\int h_x^2 dx$ as the film is very thin. This yields

$$h_t = -\left(M(h)h_{xxx}\right)_x, \tag{1}$$

when we use the equation (2) below. Here, $M(h)$ is called the "mobility term" and in thin the equation (1), $M(h) = h^n,\ \ n = 1,2,3$ is the most commonly used mobility term in the thin film theory. If E is the driving force of the evolution, then

$$h_t = \left(M(h)(\frac{\delta E}{\delta h})_x\right)_x \tag{2}$$

describes the evolution of the physical quantity $h(t, x)$. By the structure of the equation (2) the driving energy E is dissipated along the evolution defined by (2). But this is not enough to derive qualitative results on solutions of the (2). As it will be clear in our results below, finding other dissipated energy functionals are quite important in fourth order cases of (2), as there is no maximum principle for these fourth order equations, which we focus on here.

To motivate the study, we just mention some of the physical settings where thin-film evolution plays an important role. In Geology, thin-film type evolution models are employed to explain the movement of lava flows and gravity currents under water [51, 52]. In Biophysics, the thin-film dynamics appear as membranes, as tear films in the eye [89, 101], as the description of the motion of the viscous fluid in a Hele-Shaw cell [72, 77, 78] or as linings of mammalian lungs [45]. In Engineering, thin-films help in the heat and mass transfer processes, they limit fluxes and they protect surfaces [77]. The set of applications includes dynamics of paints as fluids, membranes and adhesives [72, 77]. De-icing of airplane rings and spin coating of microchips can also be regarded as two industrial processes where thin-film evolution plays a crucial role [1]. It is also worth noting that simple processes like taking shower, drinking a cup of coffee are situations where we can observe a variant of thin-film evolution. In the following sections we will briefly introduce each interesting physical process where the governing dynamics is thin-film type evolution.

We also note that in obtaining a model equation or system of equations, like (3) below, one first determines the underlying governing forces that are responsible for the evolution. By careful analysis of experiments, one determines the essential parameters and/or forces causing the evolution. In most cases the fluid film height varies due to the effect of (one or more): surface tension, gravity, surface tension gradients, viscous shear and long range molecular forces. We know from elementary fluid dynamics (see for example [1]) that the evolution of a free surface of a fluid moving down an inclined plane of angle α obeys the Navier-Stokes equations coupled with a condition on the fluid velocity. Specifically, we have

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2 u + g\sin(\alpha) - g\cos(\alpha)k, \tag{3}$$

$$\nabla \cdot u = 0, \tag{4}$$

where u is the fluid velocity, ρ is the density and μ is the viscosity which is the reason for the "friction" in the system and k is the unit normal vector perpendicular to the fluid surface. The equation (4) is called the incompressibility condition.

Navier-Stokes equations are coupled with appropriate initial and boundary conditions. As initial condition one just needs to prescribe the initial fluid profile. Boundary conditions reflect the interaction of the fluid film with the solid and with the gas at the free surface. At the fluid-solid interface, usually 'no-slip' boundary condition is used. By now, we can convince ourselves that the system of equations (3) coupled with (4) is extremely difficult to treat both analytically and numerically. By a careful analysis of the underlying particular problem, one sometimes reduces (3) to a simpler evolution equation, which still keeps the interesting dynamics of the real situation. Note that this can be checked by means of experiments and simulations. In other words, one needs to keep the balance between simplifying and keeping the characteristics of the dynamics.

In some cases, like in the thin-film type evolution processes, this is possible. Although, the equations obtained this way are much simpler, they capture the necessary information to describe the evolution process.

One important assumption in the analysis of thin-film type evolution is that the films considered are sufficiently small but still thick enough to apply the continuum theory. In other words, we are analyzing 'long-scale' phenomena only, meaning that we suppose variations along the horizontal direction are much more gradual and important than the ones normal to the film surface and they are slow in time [77]. This kind of an approach has been employed in a number of places such as, shallow-water theory for water waves, slender-body theory in aerodynamics, fiber dynamics and lubrication theory in the evolution of viscous films [77], which is the one we focus on here. It turns out that this approach simplifies (3) great deal, giving a fourth-order, degenerate, parabolic equation describing the thickness of some quantity, in our case this is thin-film thickness. The price to pay is that the resulting partial differential equation(s) usually involve a strong nonlinearity and higher-order spatial derivatives.

In thin film type evolutions, besides the complexity of the governing equations, one other important issue is to explain the physics of a moving contact line, this is a triple juncture of the liquid/air, solid/liquid, and sloid/air interfaces, the place where liquid, solid and air meet. When we use 'no-slip' boundary condition, there is a contradiction, known as the 'contact line paradox'. No-slip boundary condition requires that the parallel component of the fluid velocity at the fluid-solid interface is 0. On the other hand, the front of the fluid is obviously moving. There are several approaches to this issue, but the fluid behavior around the contact line is not yet fully understood. Inclusion of the intermolecular forces, known as the Van der Waals forces, has been a suggested way [8, 15, 83, 67]. This introduces new driving forces into the governing equations. One can also allow slip of fluid at the boundary. In recent investigations another interesting approach has been employed. It is thought that the solid surface is already wetted by a very thin layer of fluid. It turns out that this last approach is very appropriate in various situations and even in the case that the solid surface is dry but the fluid is completely wetting. An example is the spreading of silicon oil on a glass surface [56].

Before proceeding further to introduce physical processes, let us derive the general form of a thin-film type equation from an asymptotic expansion of the Navier-Stokes equations with small capillary number C_a and Reynolds number Re. We follow the approach as in [72], but we note that other derivations can be found in [56, 77] and the references therein. We consider a thin liquid surface flowing down an inclined plane of the angle of inclination α, where the driving forces are surface tension, gravity, surface tension gradients and long range molecular forces.

The first step of our analysis is the non dimensionalization process. Introduce the length scales (L_0, h_0) and velocity scales $(U, \delta U)$, where L_0 is the typical length along the film and h_0 is the typical film thickness. We note that $\delta = h_0/L_0 << 1$. In the setting at hand, the leading order terms in Navier-Stokes equations are

$$-(p+\phi)_x + u_z z + B\sin(\alpha) = 0, \tag{5}$$

$$-(p+\phi)_z - \delta B\cos(\alpha) = 0, \tag{6}$$

where ϕ is given by $\phi = \phi_0 + \frac{A}{h^3}$ and it represents the effect of the intermolecular forces. A is given by $A = \frac{A'}{6\pi\delta L_0^2 U}$, where A' is the Hamaker constant.

In (5) we choose the pressure scaling $\mu U/\delta^2 L_0$ to balance the pressure with viscous forces, which avoids a nontrivial solution in the absence of inter molecular forces in the limit $\delta \to 0$.

We let the fluid velocities to be u and v in the directions x and y respectively and we keep the continuity equation which reads

$$u_x + v_z = 0. \tag{7}$$

The number B in equations (5) and (6) is called the Bond number and is given by $B = \frac{\delta^2 \rho g}{\mu L_0}$. It measures the ratio of the gravity and viscous forces. We also note that the gravity terms are negligible either if $\delta B \cos(\alpha) = O(1)$ or $B \sin(\alpha) = O(1)$.

The associated boundary conditions are given as follows: On the free surface, $z = h(t, x)$ we assume

$$v = h_t + u h_x, \tag{8}$$

$$u_z = M\sigma_x, \tag{9}$$

$$p = -Ch_{xx}, \tag{10}$$

where M and C are constants and σ is the surface tension. On the solid substrate $z = 0$ the standard boundary condition is the classical "no-slip" condition:

$$u = v = 0, \qquad z = 0. \tag{11}$$

We will comment on the "contact line paradox" in moving contact line problem. Generally, in moving contact line problem one may assume $u = 0$ with an appropriate slip condition.

Note that ϕ is a function of x only and so that (6) can only be linear in z. Integrating this equation, one obtains

$$p = -\delta B \cos(\alpha)(z - h) - Ch_{xx}, \tag{12}$$

which is a key observation for the problem. This is the source of the higher order derivative in the problem and it simplifies the Navier-Stokes equation.

On the other hand, integrating (5) twice, and imposing boundary conditions one also has

$$u = (p_x + \phi_x - B\sin(\alpha))(\frac{z^2}{2} - hz) + M\sigma_x z \tag{13}$$

and by the continuity equation

$$v_z = -u_x \tag{14}$$

or equivalently

$$v(h) = -\int_0^h u_x dz. \tag{15}$$

The kinematic boundary condition (8), when plugged in (15), yields the governing equation for the film height as

$$h_t + Q_x = 0, \tag{16}$$

where $Q = \int_0^h u dz$ is the so-called fluid flux. Finally, plugging u in, this yields the general form of the thin-film equation

$$h_t + \left[\frac{h^3}{3}(Ch_{xxx} - \delta Bh_x\cos(\alpha) + B\sin(\alpha)) + A\frac{h_x}{h} + \frac{M}{2}\sigma_x h^2\right]_x = 0. \tag{17}$$

We note that in some settings the term $\frac{h^3}{3}$ in (17) is replaced by $\beta(h) = \frac{h^3}{3} + h^n, 0 < n < 3$. Also, it is worth noting that except for approximately horizontal surfaces, i.e. $\delta = O(1)$, the gravity term in (17) is negligible in comparison to $B\sin(\alpha)$.

Further simplification on (17) has been considered in the literature especially for analytical studies. The governing equation considered reads

$$h_t + (h^n h_{xxx})_x = 0, \tag{18}$$

where n is a parameter in the problem. Different n values represent different physical situations. $n = 1$ case is the leading order equation describing the evolution of the viscous fluid flow in a Hele-Shaw cell, $n = 2$ case is the leading order equation for the situation in which the surface tension is the driving force and not the gravity. As expected from the derivation we followed above, $n = 3$ case is the leading order equation describing the evolution of a free surface of a viscous flow under the influence of gravity. More information on thin-film equation including different derivations can be found in [19, 72, 77].

The following steady version of the general thin-film equation (17)

$$Ch^3 h_{xxx} - 3Uh = -3Uh^{(\infty)}, \tag{19}$$

known as the Landau-Levich equation, has been used widely in the literature. (19) has been first derived in [60] in the context of coating of a cinefilm when it is taken out of a bath with velocity U. It has also been used to describe the behaviour of the flow at the interface between a strip under tension and coating roller [81], the evolution of a soap film [73], and the wetting layer when a bubble moves in a capillary tube [23].

We note that (19) can be derived from (17) by a traveling wave substitution $h(t, x) = H(x - Ut)$ with an appropriate choice of integration constants $A = 0 = B$. On the other hand, in [98] (19) is derived by considering surface tension as the only driving force. The $-Uh$ term stems from the no-slip condition on a moving substrate.

Equation (19) can be considered in higher dimensions too. As the basic assumption was that in the vertical ($z-$ direction) the film is very thin, one dimensional version given in (17) can be seen as a 2-dimensional process. Indeed, one can write the analogue of (17) in higher dimensional setting as

$$h_t + \nabla \cdot \left[\beta(h)(C\nabla\nabla^2 h) - \delta B \nabla h \cos(\alpha) + Br \sin(\alpha) + A\frac{h_x}{h} + \frac{M}{2} h^2 \nabla\sigma \right] = 0. \tag{20}$$

Let us now briefly introduce the physical processes where a variant of (17) can be used to describe the evolution.

1.2 Physical Processes

1.2.1 Coating

Coating is the process of covering a surface with one or more thin layers of fluid. Simple examples include: rain running down a window and manufacturing processes such as videotape production.

In general coating is not a very easy process. One faces difficulties such as: fluid rheology may be too complex to deal with and operating conditions sometimes require a running speed and that leads to instabilities. Two examples in which instabilities occur are air entrainment and ribbing [72, 80, 85]. In the literature different coating methods have been suggested, which fall into three different categories.

(i) Free Coating : The examples are dip coating [60, 98] and spin coating, in which a fluid spreads over a spinning surface [82].

(ii) Transfer Coating : Here a uniform film is transferred into a moving substrate. Examples include : slot/slide curtain [40] and gravure coating [72].

(iii) Metered Coating : Here an obstacle is placed in order to limit the thickness of coating on the substrate. Examples are blade coating and roll coating [41, 72, 98].

There are also more recent methods of coating. Let us mention a few of them here. A different coating method is used in the paint industry to charge paint particles while spraying them onto a work piece [72]. Heating then causes the particles to produce a uniform layer. Further discussion of coating may be found in the references [7, 6, 40, 84].

If one considers a model of surface tension driven flow on a curved substrate, then (17) reduces to the form [72, 87]

$$h_t + (C\beta(h)(h_{sss} + \kappa_s))_s = 0, \tag{21}$$

where $\beta(h) = h^3 + h^n$, s is the coordinate tangential to the substrate and κ is the curvature.

1.2.2 Hele-Shaw Experiment

The Hele-Shaw cell is made of two parallel sheets of plates with a small spacing λ [52, 77, 78]. Let $H \subset \mathbb{R}^2$ denote the two dimensional cross section of a Hele-Shaw cell, so that $H \times (0, \lambda) \subset \mathbb{R}^3$ is the three dimensional gap. A viscous, incompressible fluid is squeezed between the two plates and the rest is filled with gas.

The only driving force on the fluid is surface tension, at the fluid-gas and fluid-plate interface. The three-dimensional domain filled by the fluid has a simple geometric structure; we indeed suppose that it is a cylindrical domain of the form $\Omega \times (a, b)$. Since the fluid is assumed to be incompressible the area of Ω is invariant. So the effective forces on the fluid are surface tension at the fluid-glass boundary $(\partial\Omega \cap H) \times (0, \lambda)$ and surface tension at the fluid-air boundary $\partial\Omega \cap H$. In this case, the relevant surface energy is given by

$$c_1 \lambda H^1(\Omega \cap \partial H) + c_2 \lambda H^1(\partial\Omega \cap H), \tag{22}$$

where c_1 is the surface energy per unit area of the fluid-glass interface, which may be negative, and $c_2 > 0$ is the surface energy per unit area at the fluid-air interface.

The boundary condition assumed is the "no-slip" at $\Omega \times \{0\}$ and $\Omega \times \{b\}$. The motion is described by the Navier Stokes equations (5). Navier-Stokes equations are hard to treat both analytically and numerically. If λ is much smaller than the typical length scale of Ω then the fluid motion may be well-approximated by the two dimensional Darcy-law. This leads to an easier free boundary value problem for $\Omega(t)$ that captures also the characteristics of the original problem.

This problem is surface tension driven, single phase problem. See [52, 78] for a discussion. It is worth noting that (22) is a Lyapunov functional for the dynamics under consideration. We assume that the fluid touches the glass at the lateral border, $\Omega \cap \partial H \neq \emptyset$. Young's law suggests that in the equilibrium at a contact point $x \in \partial\Omega \cap \partial H$, the contact angle θ, that is the angle between $\partial\Omega$ and ∂H, is given by the relation $c_2 \cos(\theta) = -c_1$, which imposes $|c_1| < c_2$. We assume that $\theta \in (0, \frac{\pi}{2})$, and this is due to the fact that we are interested in the partial wetting scenario and so $0 < -c_1 < c_2$ is satisfied. In the single-phase Hele-Shaw problem θ is also the dynamic contact angle, which means that it is enforced upon $\partial\Omega(t)$ throughout the evolution.

Once again the main friction source is the no-slip boundary condition at $\Omega \times \{b\}$ and $\Omega \times \{a\}$. The friction at $(\Omega \cap \partial H) \times (0, b)$ effects the Stokes flow only in a very thin boundary layer. As a consequence, only part of the two-dimensional velocity field in Darcy's law, which is normal to ∂H, needs to be non zero.

To simplify the analysis further, it is assumed that the two dimensional cross section of the region Ω of the fluid is given by the area between the graph of a function, $y = h(x) \geq 0$ and the bottom of the Hele-Shaw cell, $y = 0$. In other words, we have

$$H = \{(x, y) : y > 0\}$$

and

$$\Omega = \{(x, y) : 0 < y < h(x)\}.$$

In this case the surface energy can be written as

$$c_1|\{h > 0\}| + c_2 \int_{\{h>0\}} \sqrt{1 + h_x^2} dx, \tag{23}$$

in the free boundary problem on whole real line. In case of finite interval case one considers only an energy functional of the form

$$\int_S \sqrt{1 + h_x^2} dx$$

as the energy functional where $S = \{-a < x < a\}$, and a is a positive real number. In (23), $|A|$ denotes the one-dimensional Lebesgue measure of the set A and $\{h > 0\} = \{x : h(x) > 0\}$ is the so called wetting region. Moreover, we also have the relation

$$h_x^2 = \tan^2(\theta) \ \ on \ \ \partial\{h > 0\}.$$

As a usual assumption we are interested in the thin region where the typical vertical length scale of Ω is much smaller than the typical horizontal length scale. In other words, we assume that $(h')^2 << 1$, and so the contact angle θ satisfies $\theta << 1$. We deduce from this that this is only possible if $0 < 1 + \frac{c_1}{c_2} << 1$. Here as a general simplifying assumption it has been suggested that the surface energy functional (23) may be well approximated by the functional

$$(c_1 + c_2)|\{h > 0\}| + \frac{1}{2} c_2 \int_{\{h>0\}} h_x^2 dx \tag{24}$$

in the free boundary case in the whole real line and

$$\frac{1}{2} \int_S h_x^2 dx \tag{25}$$

in the finite interval case. One can eliminate the constants in above expressions obtaining

$$E = |\{h > 0\}| + \frac{1}{2} \int_{\{h>0\}} h_x^2 dx \tag{26}$$

and

$$E_1 = \frac{1}{2} \int_S h_x^2 dx \tag{27}$$

respectively. The motivation behind these approximations is that for small enough x, $\sqrt{1+x^2}$ can be well approximated by $1+\frac{1}{2}x^2$. Discussion of the thin-film type equations in the next section should provide a better understanding. When one substitutes the energy functional E_1 into

$$h_t = \left(M(h)(\frac{\delta E_d}{\delta h})_x \right)_x ,$$

the evolution equation in which E_d is the driving force, one obtains that

$$h_t = -(hh_{xxx})_x , \tag{28}$$

whenever $M(h) = h$. It turns out that this is the most appropriate choice for this problem. Before discussing the boundary conditions associated to (28), we note that the evolutionary free boundary problem for $\Omega(t) = \{(x, h(t, x)) : x \in \Omega\}$ has been transformed into a non-local evolution equation for $h(t, x)$. For h small(the 'small aspect regime') this equation is approximated by a local evolution equation for the rescaled $h(t, x)$. This is what is meant by the 'lubrication approximation.' We also remark that both the experimental studies and simulations [20, 31, 36, 43] show that the resulting simplifying equation captures the dynamics well enough.

Usually two types of boundary conditions are considered for the finite interval case. The fist one is the periodic boundary condition

$$\frac{\partial^i h}{\partial x^i}(\pm a) = 0, \qquad i = 1, 2, 3, \tag{29}$$

whose interpretation is the simulation of an infinite array of fluid. The next most commonly used boundary condition is "no-flux" boundary condition

$$\frac{\partial h}{\partial x}(\pm a) = 0 = \frac{\partial^3 h}{\partial x^3}(\pm a). \tag{30}$$

The Cauchy problem for the equation (28) can also be considered, i.e. now $x \in \mathbb{R}$. This free boundary value problem, introduced first by Bernis [9, 12], has been considered in [78] and in [26, 27].

1.2.3 Condensate Motion and Heat Exchangers

Condensation of thin liquid films has been used in a number of settings to transfer heat. Examples include refrigerators, cooling devices and chemical plants [72].

The main idea is to pass a vapor over metal fins that are kept below the condensation temperature. In this process, condensate builds up on the fins, flows into the channel between them and gradually drains away. It has been observed that higher flow rates decrease the transfer rate of the heat. Surface tension has been suggested to be the reason of this difference [48, 72]. Note that in this case there are locally thin films in the system that reduce the thermal resistance.

Surface tension dominates at the top of the fin. Formation of the large stresses, caused by the curvature, reduces the film thickness. In the central region the curvature on the film surface is small and the gravity dominates.

On the top or near the bottom gravity forces are negligible compared to the surface tension forces, and the governing equation for the film thickness at these places is

$$(C\beta(h)h_{xxx})_x = \frac{\gamma}{h}\Delta t, \tag{31}$$

where γ is a constant and Δt is the different of the heat between fin and vapor. The term on the right side of (31) is obtained by considering a model where build up on the fans is a multiple of Δt [72].

It has been shown that the best drainage in this region is atteined for a sharp fin tip. The geometry around the trough is equivalent to the geometry of dip coating. As a consequence, we can employ the method of [60] to determine the film thickness. Moreover, numerical results suggest that the film is very thin above the trough, by the effect of suction, and this phenomena and optimum value of the curvature at the top of the fin seem to improve the heat transfer almost three times that suggested by the Nasset theory [30, 48, 72].

1.2.4 Paint Drying

When observing a newly painted surface, one usually observes some uneven profile, such as brush marks. To produce a smooth finish one must allow a downward flow due to gravity. This is called sagging. Therefore, to have a perfect finish, one needs to keep the balance between desirable property of leveling and undesirable one of sagging.

It was suggested that [72, 75, 76] the surface tension is the reason for leveling. Although, this suggestion was useful for explaining the leveling of viscous films and certain paints, observations indicate that this is not enough for solvent-based paints. As pointed out in [72], predicted results of this method on maximum wave length of brush marks seem to contradict the observations. Moreover, some phenomena, such as "reversal" (a process causing peaks in the initial film to become troughs and vice versa), can not be explained by this approach. We refer to [72] and the references therein for details and more discussions.

Paints involving a resin in a volatile solvent were considered in [79]. A clearer explanation and generalization is given in [50, 99]. The movement of a two component, which are resin and solvent, paint can be understood by considering the following governing equations.

$$h_t + Q_x = -E_v, \tag{32}$$

$$(sh)_t + (sQ)_x = -E_v + D(hs_x)_x, \tag{33}$$

where E_v is the non dimensional evaporation rate and D is the non dimensional diffusion coefficient of the solvent, s is the solvent concentration and Q, given by

$$Q = \frac{\beta(h)}{\mu}(Ch_{xx} - \delta Bh)_x - \frac{h^2}{2\mu}Ms_x,$$

is the flux where μ is the viscosity, which is not assumed to be constant, and $\beta(h) = h^3 + h^n$ as usual. This model explains the phenomena well and can also be used for "picture framing" or fat edges. If there is no evaporation then $E_v = 0$ and (32) reduces to

$$h_t + \left(\frac{\beta(h)}{\mu}(Ch_{xx} - \delta Bh)_x\right)_x = 0, \tag{34}$$

which is a special case of (17). We refer to [58] for similar problems, defects in paint films and more discussions.

1.2.5 Marongoni Effects

We observe in our daily life that contaminated water, such as water with detergent, foams but pure water does not. The reason is that the contaminated water involves surfactants. These molecules tend to cover the fluid surface so that they reduce the surface energy or surface tension.

Surfactants stabilize the films by means of two ways [72]. The first is the Gibbs elasticity: when the film expands the concentration of surfactant decreases and surface tension increases. The second way is the Marangoni effect: before reaching the equilibrium the surface tension is higher than expected due to the movement of the molecules to the surface. Both Gibbs elasticity and Marangoni effect produce a restoring force to return the fluid to the previous state and provide film stability [3, 72].

The Marangoni effect has been employed in some industrial processes. One method of drying, known as Marongoni drying, employs the Marangoni effect and it allows one to pull out a hydrophilic surface from water [62, 66, 72]. This is used in the electronics industry to dry silicon wafers, are first etched with acid and then rinsed in a water bath. After withdrawing the wafer from the water, one observes a thin film and classical methods such as evaporation or spin drying can be employed to dry out the film. On the other hand, if one has an alcohol vapor, the film can be pulled back into water.

The basic mechanism of Marangoni drying can be observed by placing a piece of cotton wool soaked in alcohol above a thin water film which causes a flow away from the alcohol source [72]. In [74], this process has been studied, in which it was shown that the leading order problem reduces to

$$h_t = \left(\frac{1}{2} M \sigma_x h^2\right)_x, \tag{35}$$

where σ is the surface tension.

The reason why Marangoni drying is preferred is the processing speed. In the spin drying case, in the final stages the film thickness decays exponentially fast, but for the final layer evaporation should be employed which takes longer time than the equivalent Marangoni system. Moreover, Marangoni drying results in a clean surface. We refer to [72] and the references therein for further discussions.

1.2.6 Long Range Molecular Forces

After reaching an equilibrium thickness, a thinning film reaches a critical value after which film rupture occurs in a very short time-scale [72]. Since the length scale is so small, long range molecular forces become extremely important. These forces have been suggested to explain the rapid film rupture in the literature [86, 96]. To model the London Van der Waals force, which is an attractive intermolecular force that tries to thin the film, a potential energy functional over a unit liquid volume

ϕ has been suggested in [83], introducing a body force $F = -\nabla\phi$ in the Navier-Stokes equations. The governing equation then reads

$$h_t + \left(\beta(h)h_{xxx} + A\frac{h_x}{h}\right)_x = 0 \tag{36}$$

for a system in which surface tension and long range molecular forces are the only driving forces.

There has been some work on (36) in the literature [72, 83, 67]. The question whether or not a small free surface perturbation grows or decays in the presence of intermolecular forces has been investigated in [83]. In [67], the influence of non linear perturbations on the equation (36) has been investigated to check the validity of the linear theory.

On the other hand, observing that the term $(\frac{h_x}{h})_x$ is unrealistic at a microscopic length-scale Bertozzi and Pugh [15] considered

$$h_t + (h^n h_{xxx})_x + (h^m)_{xx} = 0, \tag{37}$$

which describes the evolution of the scaled film height around a contact line. Both analytical and numerical methods have been employed in [15] to show that finite time film rupture occurs in (37) and that the front has a finite speed of propagation.

1.2.7 Foams and Free films

By a dry foam we mean a two-phase fluid where a small amount of gas is separated by a thin film. The film is continuous and the volume fraction of the liquid is small. The films considered in this case have negligible thickness and liquid content, and most of the liquid is in the Pletaue borders or in the film junctions [72].

Lubrication theory applied to free films can cause problems. The reason is that zero shear on the free surface and symmetry at the center will give two conditions for a single arbitrary constant in the expression of the velocity gradient. Two methods have been proposed to overcome this difficulty [72]. The first one is to consider a higher order approximation as in (32) or the second idea is to assume that the surface is saturated with surfactant. This implies that the film is inextensible to have constant velocity. We refer to [72] and the references therein for further discussions.

1.2.8 Power Law Energy Functionals

On the other hand, another example is the problem of relaxation of axisymmetric crystal surfaces with a single facet below the roughening transition. In [65] (also the references therein) this problem is analyzed via a continuum approach that accounts for step energy g_1 and step-step interaction energy $g_2 > 0$. We point out that the evolution of the surface morphology here is caused by the motion of steps. The energy functional used for this problem is:

$$H_3(h) := \int \left(g_0 + g_1|\nabla h| + \frac{1}{3}g_2|\nabla h|^3 \right) dx, \tag{38}$$

where g_0 term represents the surface free energy of the reference plane, g_1 is the step energy and g_2 includes entropic repulsions due to fluctuations at the step edges and pairwise energetic interactions between adjacent steps. We will omit details and moreover we will not analyze the equation obtained closely. We mention this problem to show that there are situations in which different power law surface energy functionals are used.

The above problem, together with the observation that the other power law energy functionas may be used in the thin film equation motivates the study of the following family of equations, which we work in this thesis.

$$h_t = -\left(h^n \left((p-1)(h_x^2)^{\frac{p}{2}-1} h_{xx}\right)_x\right)_x . \tag{39}$$

Note that $p = 2$ case in (39) reduces to the classical thin film equation case. Details of our results on the equation (39) will be stated in the upcoming chapters.

There are probably other places and physical processes in which thin film type of evolution plays an important role. We remark that spreading of a liquid on a solid is important but this process is poorly understood.

Let us comment on the difficulties and unsolved interesting issues. The idea in the Laplace-Young equation is that the energy must be stationary with respect to any shift dx of the line position [35]. Moreover, here θ_e, the apparent contact angle, is entirely defined in terms of thermodynamic parameters: by which we mean that the measurement of θ_e gives us certain information on the interfacial energies. Usual approaches in determining the angle θ_e are: direct photograph, through the reflections or deflection of rays by the liquid prism, by interferential techniques or from the rise of liquid column in a fine capillary [35]. The experimental difficulty here is that one wants to avoid a certain "pinning of the triple line L" on defects of the solid surface. This causes a hysteresis of the contact angles, which can seriously affect the determination of θ_e. On the other hand, the contact line L itself can be curved and in this case a displacement of the line modifies the core energies. This leads to measurable effects only when the radius of curvature of the line is not too large, compared to the core size. A measurement of θ_e at a distance d from the contact line should give a well-defined θ_e, independent of d.

In non equilibrium situations, one may have a solid/gas interfacial energy which gives rise to so called spreading coefficient

$$S = \gamma_{so} - \gamma_{sl} - \gamma,$$

where γ_{so} is the energy associated with a "dry" solid surface. Importance of spreading coefficient has been first pointed out by Cooper and Nuttal [32, 35] in connection with the spreading of the insecticides on leaves. $S > 0$ large implies spreading of liquid. But there is an ambiguity when experimentalists observe complete spreading on macroscopic scales: they can not tell whether $S = 0$ or $S > 0$ [35]. We also note that the smaller S is the larger is the equilibrium thickness [35], which agrees with results of Cooper and Nuttal [32], [35]. θ_e depends chemical constitution of both the solid and liquid, [102].

To this end we note that solids can be categorized in two ways: high-energy surfaces and low-energy surfaces [35]. Also we can say that there are two types of solids: hard solids, such as covalent, ionic or metallic, and weak molecular crystals, such as molecules bound by hydrogen bonds or molecules bound by Van der Waals forces [35]. We also note that high-energy surfaces are wetted by molecular liquids, not because γ_{so}, the energy associated with a "dry" solid surface, is high, but rather because the underlying solid usually has a polarizability p_s much higher than the polarizability of the liquid. This approach is very primitive but it provides us a guidance.

Low-energy surfaces, on the other hand, can give rise to both partial or complete wetting depending on the liquid [35]. When plotting θ_e as a function of the surface tension γ of the liquid we expect that there is a critical value γ_c for which θ_e is 0. Moreover, we also expect that such a value should depend not only on the solid but also the liquid [35] but Zisman [102] observed that γ_c is independent of the nature of the liquid, and is a characteristic of the solid alone. Whenever we want to have a molecular liquid wetting a low-energy surface we must choose a liquid with surface tension $\gamma < \gamma_c$. Various authors have tried to to relate the critical value γ_c to some physical parameters of the solid [35] (and the references therein) and the conclusion is that it is an increasing function of the polarizability and moreover high γ_c surfaces, such as nylon, PVC are most wetted surfaces by liquid [35].

Once again, the "contact line" paradox is the problem that has been in the center of recent investigations in physics literature [38, 37, 63, 57, 71, 90]. Additional to the mentioned approaches Barenblatt, Beretta and Bertsch in [4] have proposed dividing the free surface of the liquid into three regions. These regions are called basic region, contour region and precursor region [4]. In the basic region the regular lubrication approximation is valid but in the contour region the film is not necessarily gently slopping and the Laplace formula is not valid due to the nonequilibrium of the character of the distribution of the cohesive forces.

1.3 Mathematical Side of the Story

As mentioned at the beginning, most of the analytical studies focus on the equation

$$h_t + (|h|^n h_{xxx})_x = 0, \tag{40}$$

where $n > 0$ is a parameter and on physical grounds $h \geq 0$ and so the following equation replaces (40)

$$h_t + (h^n h_{xxx})_x = 0. \tag{41}$$

Before introducing our accomplishments on the subject, let us review mathematical results and techniques on thin-film type equations.

1.3.1 Asymptotic Results

The basic question here is to determine the steady film thickness. To this end, the method of [60] has been used in some other situations [23, 48, 73, 88]. Far from the bath, used in dip coating, the film has an unknown constant thickness, $h^{(\infty)}$, and in the neighborhood of the bath the thin-film approximation is invalid [72]. Thus, one may employ the Laplace-Young equation to determine the film shape, or another approach is to assume that the film is a circular arc, and finally one must match these two regions onto the transition region where the Landau-Levich equation (19) determines the film thickness [72].

In the same manner, asymptotic expansion techniques have been applied to drop spreading problem [49, 59]. Here, one needs to consider three regions if a slip term is included. As in [4], these are: an inner region where the film height is of the same order as the slip length, the outer region where one can apply the lubrication approximation and which describes flow in the bulk. Finally, there is also an intermediate region where an expansion in terms of a slip length is required [49, 59, 91].

There are also other situations in which this kind of an approach has been employed [23, 48, 68, 73, 70, 88, 92, 98].

1.3.2 Similarity Solutions

Usually in thin-film type evolution equations leading order behavior describing types of singularity and large time behavior are important issues. Similarity solutions have been suggested to represent these behaviors [18, 21, 72].

To describe the singularity formation one may define

$$h(t, x) = T(t)H(\frac{x - \bar{x}(t)}{T^q(t)}) = T(t)H(\eta), \tag{42}$$

where $\bar{x}(t)$ is the position of the minimum film thickness. The functions T and H should satisfy [72]

- $T(t) \to 0$ as the singular time is approached,
- $H > 0$
- H is well-behaved for η large enough, and this allows matching at the boundaries.

Plugging (42) into (41) yields an ODE of the form

$$\frac{T_t}{T}(1 - q\eta\frac{\partial}{\partial\eta})H - \frac{\dot{\bar{x}}}{T^q}H_\eta + T^{n-4q}(H^n H_{\eta\eta\eta})_\eta = 0. \tag{43}$$

In this form, (43) is complicated to treat analytically and so it may be simplified either by choosing $\bar{x} = \alpha T^q$ or by choosing various terms dominate [72]. Numerical work on (43) has also been performed [21] to analyze the singularity formation in the thin-film equation.

Source type solutions to thin film equation (41) are solutions with initial data

$$h(0, x) = h_0 \delta(x), \tag{44}$$

where h_0 is the mass and $\delta(x)$ is the Dirac delta function at x. In this case, one may employ the similarity variables and use the dimensional analysis to obtain [13, 10, 72, 91]

$$h(t, x) = \frac{1}{t^k} H(\frac{x}{t^k}) = \frac{1}{t^k} H(\eta), \tag{45}$$

where $k = \frac{1}{n+4}$. Plugging this in (41) and integrating the resulting equation, by using the symmetry property of the source type solutions, one gets

$$H^{n-1} H_{\eta\eta\eta} = c\eta.$$

In [91], the exact solutions of this equation in the cases $n = 0, 1$ are found under the condition that the solutions are symmetric at the origin and moreover at the contact line $\eta = \eta_*$

$$H(\eta_*) = H'(\eta_*) = 0.$$

Without showing that the series converges, the authors in [91] have obtained approximate solutions using a Frobenious-type series expansion in the case $\frac{3}{2} < n < 3$. The exact form of the similarity solution for $n = 1$ case is

$$v^{(\infty)}(x) = \frac{1}{24}(C^2 - x^2)_+^2, \tag{46}$$

where g_+, as usual, is given by $g_+(x) = \max(0, g(x))$ and the constant C is determined through the mass conservation.

In [13] and [10] similarity solutions of the thin-film equation (41) have been studied. Under the same assumptions as [91], it was shown that compactly supported, non negative source type solutions exist if and only if $n < 3$. In the case $3/2 < n < 3$, this is easy to see: as $\eta \to \eta_*, H \to$

$(\eta_* - \eta)^\beta$ [10] and plugging this into the ODE and linearizing around $\eta = \eta_*$ yields $\beta = \frac{3}{n}$, and $H_\eta = 0$ implies $n < 3$.

Recently, similarity variables have been employed in [64], [27]. The idea here is to use a time dependent change of variables to obtain a new equation that has a unique steady-state. Using the ideas of the Kinetic theory, which is applicable if the system has a unique steady state, then the convergence of an appropriate Lyapunov functional shows that solutions of the rescaled equation converge eventually to the unique steady state. Using a Csiszár-Kullback type inequality this information is then translated into the convergence of solutions to the unique steady state under an appropriate norm and the time scaling then provides the convergence of solutions of the original equation to the self-similar solutions. We also employ such a study which we will explain in detail in later sections.

Various numerical studies [21, 24, 70, 69, 87, 100] and stability analyses [18, 24, 29, 53, 99] have been done on the thin-film equation (41), which we do not mention in detail. There are also very recent investigations in this direction. Basic questions one is concerned with in numerical studies are [72]

- whether $h \to 0$ in finite or infinite time
- whether $h \to 0$ in an interior point or at the boundary
- whether solutions have reflective symmetry about the singular point.

1.3.3 Partial Differential Equations Techniques

We now provide the main PDE techniques on the thin film type equations. The first work to this end can be taken as the excellent work of Bernis and Friedman [14] in which most of the physical questions have been answered in mathematically rigorous ways. Let us briefly introduce the methods of their analysis. The first step is to provide a "weak solution" concept as we do not expect to have four derivatives of $h(t,x)$ around the vicinity of the contact line. Hence, the following weak solution concept has been introduced [14]

Definition : Let $\Omega = (-a,a), a > 0$ be a finite interval. A non negative function $h \in C^{1,4}_{t,x}(\{h > 0\}) \cap C^{1/8,1/2}_{t,x}(Q_T)$ that satisfies $h^{n/2}h_{xxx} \in L^2(\{h > 0\})$ and $h_x = h_{xxx} = 0$ on $\partial\Omega$ is a weak solution of (41), where $Q_T := (0,T) \times \Omega$, $\frac{1}{p} + \frac{1}{p'} = 1, \beta = \frac{p-1}{5p-2}$ and $f \in C^{a,b}_{t,x}$ means that the function f is Hölder continuous of order a and b in t and x respectively, if

$$\int_0^\infty \int_\Omega (h - h_0)\zeta_t dxdt + \int_0^\infty \int_\Omega h^n h_{xxx}\zeta_x dxdt = 0, \tag{47}$$

for all $\zeta \in C^1(\mathbb{R}^+ \times \bar{\Omega})$ such that $\zeta(T, \cdot) = 0$.

We note that this weak formulation given in the above definition is obtained from (41) by multiplying by ζ and integrating by parts. In this way the differentiability requirement on solutions may be weakened.

By showing that the solution of the regularized problem

$$h_t + ((h^n + \epsilon)h_{xxx})_x = 0, \tag{48}$$

for $\epsilon > 0$, are Hölder continuous with respect to x and t uniformly in ϵ, they show that the sequence $\{h_\epsilon\}$ is bounded and equicontinuous so that by the Arzelá-Ascoli Theorem there exists a function h so that $h_\epsilon \to h$ as $\epsilon \to 0$. Moreover, they show that h is a weak solution of the thin film equation (41) in the sense given in the above definition.

Now, for physical ground to make sense it needs to be shown that

$$h_0 \geq 0 \implies h(t,x) \geq 0, \qquad \forall t > 0, \forall x \in \Omega,$$

which is shown in [14] by employing entropy dissipation methods. To explain, one defines an entropy functional of the form

$$G_\epsilon(s) = -\int_s^A g_\epsilon(r)dr,$$

where $A > \max(|h_\epsilon|)$ and $g_\epsilon(s) = -\int_s^A \frac{1}{f(r)+\epsilon}dr$ and they show that $G_\epsilon(\cdot)$ is dissipated along any solution of regularized equation (48). For $1 < n < 2$ no restriction on the initial data is required besides $h_0 \geq 0$, for $n = 2$ one needs to assume $\int_\Omega |\log(h_0)|dx < \infty$ and for $2 < n < 4$ one needs $\int_\Omega h_0^{2-n}dx < \infty$ and for $n \geq 4$, $h_0 > 0$ in $\bar{\Omega}$. We remark also that any weak solution of (41) with $n \geq 4$ is positive and so classical.

In addition, one may ask how the support of solutions behave. Such a question is first addressed in [14] by showing that the support of the function $t \to h(t, \cdot)$ is increasing in time t. The positivity result ($n \geq 4 \implies h(t,x) > 0$ if $h_0 > 0$) motivated another regularization where one defines

$$f_\epsilon(s) = \frac{s^4 s^n}{\epsilon s^n + s^4}, \tag{49}$$

$h_{0\epsilon}(x) = h_0(x) + \epsilon^\theta, 0 < \theta < 1/2$ and considers

$$\begin{aligned} h_{\epsilon,t} + (f_\epsilon(h_\epsilon)h_{\epsilon,xxx})_x &= 0, \\ h_\epsilon(0,x) &= h_{0\epsilon}(x). \end{aligned} \tag{50}$$

A new weak solution h is defined by the limit $h_\epsilon \to h$ as $\epsilon \to 0$, where h_ϵ is the positive smooth solution of the equation (50). By defining the analogs of g_ϵ and G_ϵ by

$$g_{1\epsilon}(s) = -\int_s^A \frac{dr}{f_\epsilon(r)}, \qquad G_{1\epsilon}(s) = -\int_s^A g_{1\epsilon}(r)dr$$

one establishes exactly the same kind of nonnegativity results.

We also note that higher-order equations of the form

$$h_t + (-1)^{m-1}\left(f(h)\frac{\partial^{2m+1}h}{\partial x^{2m+1}}\right)_x = 0, \tag{51}$$

where $f(s) = |s|^n f_0(s)$ with $f_0(s) > 0, n \geq 1$, were also introduced in [14] and similar results are also mentioned for these equations too.

After the excellent study of Bernis and Friedman [14], many authors have tried to extend the results of this paper and in particular in [16] the weak solution concept has been developed. Interestingly, in [21] a similar entropy functional like $G_{1\epsilon}(\cdot)$ defined above to improve the positivity result which lead to: if $n \geq 3.5$ then there is no singularity formation of the form $h \to 0$. Let us mention the results of [16] briefly. The authors in [16] show that there exists a non negative weak solution for $0 < n < 3$ where they consider two types of solutions under periodic boundary conditions, whose interpretation is modeling a periodic array of droplets. The first problem has initial data $h_0(x) \geq 0$, for $0 < n < 3$ in (41). It has also been shown in [16] that there exists a finite time $T^* \geq 0$ so that after this time on a weak solution becomes a positive, strong solution and moreover $h \to \frac{1}{|\Omega|}\int_\Omega h(t,x)dx$ as $t \to \infty$. These weak solutions are in the classical sense of distributions for $\frac{3}{8} < n < 3$ and in the sense of [14] for $0 < n \leq \frac{3}{8}$. Regularity of these solutions have also been established so that this weak solution concept just includes the unique source type solutions [10] with 0 slope at the edge of the support and they do not include any less regular solutions with positive slope at the edge of the support [16].

Also strictly positive initial data $h_0(x) > 0$ of (41) for $0 < n < \infty$ has been considered in [16]. It is remarkable result of [16] in this case that even if a finite time singularity of the form $h \to 0$ occurs there exists a non negative weak solution for all time. And same type of results mentioned in above paragraph are established.

As one may expect, the main technical idea of [16] is to introduce new classes of dissipating entropies to prove existence and higher regularity of these weak solutions. The long time behavior has been established by showing that the entropy is related to norms of the difference between the solution and its mean value, which is sort of Csiszár-Kullback type inequality.

Independently, around the same time, [8] establishes results on support properties, long-time behavior and higher regularity of non negative weak solutions. As usual, the main ingredient of the analysis here is also the dissipation of a certain family of entropies. Indeed, they define the following functions [8]

$$g_{2\epsilon}(s) = -\int_s^A \frac{\alpha r^{\alpha+n-1}}{f_\epsilon(r)} dr,$$

$$G_{2\epsilon}(s) = -\int_s^A g_{2\epsilon}(r) dr,$$

where $A > \max(h_\epsilon)$ and $0 < s < A$. Note that as $h_\epsilon > 0$, where h_ϵ is the unique smooth solution of the regularized equation (50) and f_ϵ is given in (49), $g_{2\epsilon}, G_{2\epsilon}$ are well-defined functions. By a careful analysis, the authors prove that $G_\epsilon(h_\epsilon)$ dissipates and in particular they deduce an integral estimate for h_ϵ. Furthermore, it is shown in [8] that this also holds for non negative weak solution obtained as a limit of the sequence $\{h_\epsilon\}$ as $\epsilon \to 0$. It turns out that these estimates are quite useful in obtaining the following qualitative results on weak solutions, which we briefly introduce.

- $n \geq 2 \implies supp(h(t_1, x)) \subseteq supp(h(t_2, \cdot))$ for $0 \leq t_1 \leq t_2$ and

  $$n \geq 4 \implies supp(h(t, x)) = supp(h_0(\cdot)), \qquad t \geq 0.$$

  $n \geq 7/2$ and $h(t_0, x_0) > 0 \implies h(t, x_0) > 0, \qquad t > t_0.$

  $n \geq 2$ and $h(t_0, x_0) > 0 \implies h(t, x_0) > 0$ almost every $t > t_0$.

- $0 < n < 3 \implies h(t, \cdot) \in C^1([-a, a])$ almost every $t > 0$, and h becomes strictly positive after some finite time and

  $h(t, x) \to \frac{1}{2a} \int_{-a}^{a} h_0(x) dx$ uniformly in $[-a, a]$.

Here $supp(f)$ is the support of the function f. They also provide a counter example for the finite time rupture phenomena [8]. We use the ideas of [8] in the analysis of one of the equations we derive and details of these calculations are provided in upcoming sections.

One can summarize these entropy estimates by writing: for $\alpha \in (\max\{-1, \frac{1}{2} - n\}, \{2 - n\}) - \{0\}$ there holds the inequality

$$\begin{aligned} \frac{1}{\alpha(\alpha+1)} \int_\Omega h^{\alpha+1}(T, \cdot)dx &+ \frac{1}{C} \int_0^T \int_\Omega (|\nabla h^{(\alpha+n+1)/4}|^4 + |D^2 h^{(\alpha+n+1)/2}|^2) dxdt \\ &\leq \frac{1}{\alpha(\alpha+1)} \int_\Omega h_0^{\alpha+1}(x)dx. \end{aligned} \quad (52)$$

Note that this is written in the higher dimensional setting

$$\begin{aligned} h_t + div(h^n \nabla \Delta h) &= 0, \\ h(0, x) = h_0(x) & \end{aligned} \quad (53)$$

with either periodic or "no-flux" boundary conditions. The higher dimensional extension of the thin film equation has also been studied in the literature and similar results as in the one dimensional case have also been established [22, 33, 46, 47].

We note that there is no physical interpretation of the estimate given in (52) whereas the estimate

$$\frac{1}{2} \int_\Omega |\nabla h(T, \cdot)|^2 dx + \int_0^T \int_\Omega h^n |\nabla \Delta h|^2 dxdt = \frac{1}{2} \int_\Omega |\nabla h_0|^2 dx \quad (54)$$

has physical meaning and is called the "energy estimate." As pointed out before, the first term on the left in (54) is the linearized capillary energy in the complete wetting region. The second term on the left indicates that the energy is dissipated due to the viscous friction. We note that both the stationary and non-stationary solution to initial data $h_0(x) = m(A^2 - x^2)_+^2$ satisfy the energy estimate but only the non-stationary zero contact angle solution satisfies the entropy estimate [5].

After these excellent studies [8, 14, 16], more attention on fourth-order equations, as the equation (41), has been provided in the mathematics community. Rigorous studies emerged from this curiosity, which lead to significant improvement in this field. Nevertheless, there are still interesting open issues, some of which have been addressed pretty recently.

When one has a partial differential equation associated to an evolution phenomena, one is concerned if the model is physically correct one. One of the characteristics of the physically correct free boundary value problem is the finite speed of propagation. In the thin-film case the question is that whether or not the Cauchy problem

$$\begin{aligned} h_t + (h^n h_{xxx})_x &= 0, \qquad x \in \mathbb{R}, t > 0 \\ h(0,x) &= h_0(x), \qquad x \in \mathbb{R} \end{aligned} \tag{55}$$

has finite speed of propagation. Bernis has addressed this issue in two consecutive papers [18, 19]. In the first one [19] he proved that for $0 < n < 2$ the problem (55) has a finite speed of propagation for non negative strong solutions and thus there is an interface or free boundary separating the regions $h > 0$ and $h = 0$. Then he observes that the interface is Hölder continuous if $\frac{1}{2} < n < 2$ and is right-continuous if $0 < n < \frac{1}{2}$. Moreover, he studies the Cauchy problem and obtains optimal asymptotic rates as $t \to \infty$ for the solution and for the interface when $0 < n < 2$, which exactly match those of the source-type solutions. For $0 < n < 1$ the property of the finite speed of propagation is also provided [19]. In the consecutive paper [18], he also provides that the equation (55) has a finite speed of propagation and that the interface is Hölder continuous for the remaining case $2 \leq n < 3$.

When $n = 1$, the Cauchy problem (55) has a deeper connection to so called *Wasserstein distance.* To be precise let us recall the definition.

Definition [78] : For given $\rho_0, \rho_1 \in K$, where

$$K := \left\{ \rho : \mathbb{R} \to [0, \infty) : measurable, \int_{\mathbb{R}} \rho dx = 1, \int_{\mathbb{R}} x^2 \rho(x) dx < \infty \right\},$$

which is the set of all configurations with given unit volume, introduce the space $P(\rho_0, \rho_1)$ of admissible "transference plans" defined as

$P(\rho_0, \rho_1) := \{p-$non negative Borel measure on $\mathbb{R} \times \mathbb{R} : \int_{\mathbb{R}\times\mathbb{R}} \zeta(x) p(dxdy) = \int_{\mathbb{R}} \zeta(x) \rho_0(x) dx,$
$\int_{\mathbb{R}\times\mathbb{R}} \zeta(y) p(dxdy) = \int_{\mathbb{R}} \zeta(y) \rho_1(y) dy, \forall \zeta \in C_0^0(\mathbb{R})\}.$

The *Wasserstein distance* between ρ_0 and ρ_1, denoted by $W_2(\rho_0, \rho_1)$, is defined by the relation

$$W_2^2(\rho_0, \rho_1) := \inf_{p \in P(\rho_0, \rho_1)} \int_{\mathbb{R}\times\mathbb{R}} |x - y|^2 p(dxdy). \tag{56}$$

We say that $P(\rho_0, \rho_1)$ is the set of probability measures on $\mathbb{R}\times\mathbb{R}$ with first marginal $\rho_0 dx$ and the second marginal $\rho_1 dy$ and that this set always contains the product measure $\rho_0(x)dx\times\rho_1(y)dy$ by the bounds of the second moments. Brenier [11] has shown the uniqueness of the optimal transference plan and moreover he also showed that its support is concentrated on the graph of the gradient of a convex function.

By considering a variational scheme Otto [78] proved the long-time existence of a weak solution $h(t, x) \geq 0$ of the thin-film equation (55) in the region $\{h > 0\}$ with prescribed contact angle $\frac{\pi}{4}$, meaning

$$(h_x)^2 = 1 \text{ on } \partial\{h > 0\}.$$

To this end we recall the variational scheme.

Variational Scheme [78] :

We fix a time step $\tau > 0$, and consider sequences $\{h^{(k)}\}_{k\in\mathbb{N}} \subset K$ which satisfy
$h^{(k)}$ minimizes $\frac{1}{2}W_2^2(h^{(k-1)}, h)^2 + \tau E(h)$ among all $h \in K$ for all $k \in \mathbb{N}$ and we set $h^{(0)} := h_\tau^0$, where E is given by $E = \int h_x^2 dx$. And the non negative weak solution of (55) is obtained by the limit $h_\tau^{(k)} \to h$ as $k \to \infty$. We refer to [78] for details.

In [33], higher dimensional case of the thin film equation has been studied. By means of energy and entropy estimates, analogs of one dimensional case, they prove existence and positivity results in higher space dimensions for the equation (53) with non negative initial data. They also discuss asymptotic behavior for $t \to \infty$ of solutions and provide a counter example to the issue of uniqueness. Grün [44] considers the same problem and solves the Cauchy problem for $n \in [2, 3)$. He observes that the new interpolation inequalities applied to the existing energy estimate controls the third order derivatives of appropriate powers of solutions. In such a way, he extends the solution concept of Bernis and Friedman [14] to multi-dimensional setting. Moreover, he also provides a key integral estimate to deduce results on the qualitative behavior of solutions such as "finite speed of propagation" or "occurrence of a waiting time phenomena."

J.R. King introduces two generalizations of the thin film equation in [54], the first of which reads as

$$h_t = -\left(h^n h_{xxx} + \alpha h^{n-1} h_x h_{xx} + \beta h^{n-2} h_x^3\right)_x, \tag{57}$$

which shares the same scaling properties of the thin film equation (41) and arises in applications. The second one reads as

$$h_t = -\left(h^n |h_{xxx}|^{m-1} h_{xxx}\right)_x, \tag{58}$$

called as the "doubly-nonlinear thin film equation" and is relevant to capillary driven flows of thin liquid films of power-law fluids. The author in [54] gives a characterization of non negative mass preserving compactly supported solutions, exploits local analyzes about the edge of the support and obtains special closed form solutions. Also he notes other properties and mentions some open problems [54].

Ansini and Giacomelli [2] prove existence of solutions to the problem (58) and obtain sharp upper bounds for the propagation of their support. They also derive a necessary condition for the occurrence of waiting-time phenomena [2], a recently investigated issue in the literature [42].

Recent work of Giacomelli and Grün [42] deals with finding a lower bound on waiting times for degenerate parabolic equations and systems, which in particular includes the thin-film equation (41). Here, they extend the method of [34] to obtain qualitative estimates on waiting times for free boundary problems.

The art of finding new appropriate energy functionals for fourth-order equations like (41) is extremely important tool in analyzing these equations. To this end, Laugesen [61] has introduced new dissipated energies for the thin film equation (41). In fact, he showed that the energy $K_q := \int \frac{h_x^2}{h^q} dx$ is dissipated for positive smooth solution of (41) for some values of $q \neq 0$ when $\frac{1}{2} < n < 3$ We also employ this energy functional in our analysis. In particular, we ask whether or not an inequality of the form

$$\frac{d}{dt} K_q(h(t,x)) \leq -\Phi(K_q(h(t,x))), \tag{59}$$

with Φ some strictly positive monotone increasing function on $\mathbb{R}_+$ hold for a positive smooth solution of the equation (41) and (30). It turns out that for the physical cases $n = 1$ and $n = 2$, one can prove an inequality of the form (59) for a positive smooth solution of the equation (41) and (30).

An interesting recent study of Tudorascu [93] employs the Dirichlet energy $E = \int h_x^2 dx$ as a Lyapunov functional and uses it to deduce long-time behavior of both positive smooth solutions and nonnegative weak solutions. The idea, which we also employ in some parts of our analysis here, is to bound the energy production term $\int h^n h_{xxx}^2 dx$ in terms of the energy functional E from below and this yields an inequality of the form (59). Interestingly, the author proves that the inequality (59) is preserved under the regularization (49) and in the limit $\epsilon \to 0$.

We also note that in [15], the authors discuss a physical justification for the presence of a "porous media term" when $n = 3$ and $1 < m < 2$ in the equation

$$h_t = -(h^n h_{xxx} - (h^m)_x)_x. \tag{60}$$

They proposed such behavior as a cut-off of the singular "disjointing pressure" modeling long range Van der Waals forces, which was mentioned in the discussion of physical processes. For all $n > 0, 1 < m < 2$, the authors in [15] also discuss possible behavior of solutions at the edge of the support by employing leading order asymptotic analysis of traveling wave solutions. Rigorous weak existence theory for (60) has been presented for $n > 0, 1 < m < 2$. The presence of second order term in (60) leads to non negative weak solutions that have additional regularity. Similar to [16], they show that there is a finite time T^* after which weak solutions become positive strong solutions and they eventually relax to their mean value. Several numerical calculations presented in [15] suggest that the weak solutions described by this developed theory has compact support with finite speed of propagation. Note that these are properties desirable for a physically correct model [15].

CHAPTER II

BRIEF INTRODUCTION OF OUR RESULTS

2.1 Introduction

There has been much investigation of higher order nonlinear degenerate equations of the form

$$h_t = \left(M(h)\left(\frac{\delta H}{\delta h}\right)_x\right)_x, \tag{61}$$

where M is a specified function and H is the quadratic first order energy functional $\frac{1}{2}\int h_x^2 dx$. The energy functional arises in many physical models, but is not universal among higher order parabolic equations. Recent investigations have motivated the study of other energy functionals, such as $H_p = \int (h_x^2)^{p/2} dx$ for $p \neq 2$. We undertake such a study here, proving existence of weak solutions for appropriate boundary conditions, nonnegativity, and positivity properties of solutions. Moreover, an entropy dissipation- entropy estimate for solutions of this equation is obtained. Support properties and long time behavior of solutions are also discussed for various cases. In the next section we will briefly introduce these results after which we also state the precise description of our results on the Cauchy problem (95).

It is worth noting that the effective interface Hamiltonian taken in the derivations of the thin-film equation is an approximation. In the physical problem we are interested in the two dimensional cross section Ω of the fluid is given by the area between the graph of a function $y = h(x) \geq 0$ and by $y = 0$. Hence, $\Omega = \{(x, y) : 0 < y < h(x)\}$. Note that in this framework, the surface energy can be written as

$$H_s = \int_{\{h>0\}} \sqrt{1 + h_x^2} dx, \tag{62}$$

where we neglected the multiplicative factor [78]. From this, the approximate energy functional $\frac{1}{2}\int h_x^2 dx$ is obtained. This is because in the classical lubrication approximation the basic assumption is that the typical length scale in vertical direction is negligible compared to the typical horizontal length scale. We also neglected the constant term which does not contribute to (61).

The thin-film equation is $p = 2$ case of the “doubly nonlinear thin film equation” [2]:

$$h_t + \left[|h|^n |h_{xxx}|^{p-2} h_{xxx}\right]_x = 0, \tag{63}$$

where $n > 0$ and $p \geq 2$ are real constants. Equation (63) describes the evolution of the height $h(t, x)$ of a surface-tension driven thin liquid film on a solid surface in lubrication approximation [2], [55], [77], [97]. $p = 2$ case in (63) corresponds to a Newtonian fluid, and $p \neq 2$ occurs when considering “power-law” liquids. In [2], the authors prove the existence of solutions to the problem (63), and obtain sharp upper bounds for the propagation of their support. They also derive a necessary condition for the occurrence of waiting-time phenomena.

Another example is the problem of relaxation of axisymmetric crystal surfaces with a single facet below the roughening transition. In [65](and the references therein) this problem is analyzed via a continuum approach that accounts for step energy g_1 and step-step interaction energy $g_2 > 0$. The evolution of the surface morphology here is caused by the motion of steps. The energy functional used for this problem is:

$$H_3(h) := \int \left(g_0 + g_1|\nabla h| + \frac{1}{3} g_2 |\nabla h|^3 \right) dx, \tag{64}$$

where g_0 term represents the surface free energy of the reference plane, g_1 is the step energy and g_2 includes entropic repulsions due to fluctuations at the step edges and pairwise energetic interactions between adjacent steps. We will omit details and moreover we will not analyze the equation obtained closely. We mention this problem to show that there are situations in which different power law surface energy functionals are used.

To facilitate the reading we will briefly introduce our results in next two sections. The precise statements and proof techniques of these results are also briefly mentioned to give the flavor of the detailed proofs of the upcoming chapters.

2.2 Brief Introduction of Our Results on a New Family of Degenerate Parabolic Equations

With the given background above as motivation, we now turn to the study of (61) for $H = H_p$, where H_p is given by (65) with $p > 0$,

$$H_p(h(t,x)) := \frac{1}{p}\int |h_x(t,x)|^p dx. \tag{65}$$

A simple set of calculations yields that

$$\frac{\delta H_p}{\delta h} = -(p-1)(h_x^2)^{\frac{p}{2}-1}h_{xx},$$

and differentiating this with respect to x yields

$$(\frac{\delta H_p}{\delta h})_x = -(p-1)(p-2)(h_x^2)^{\frac{p}{2}-2}h_x h_{xx}^2 - (p-1)(h_x^2)^{\frac{p}{2}-1}h_{xxx}.$$

Plugging this back into (61), with $M(h) = h^n$, yields:

$$h_t = -[h^n((p-1)(p-2)(h_x^2)^{\frac{p}{2}-2}h_x h_{xx}^2 + (p-1)(h_x^2)^{\frac{p}{2}-1}h_{xxx})]_x. \tag{66}$$

Therefore the initial boundary value problem we consider here is

$$h_t = -[h^n[(p-1)(h_x^2)^{\frac{p}{2}-1}h_{xx}]_x]_x, \tag{67}$$

in $Q_T := (0,T)\times\Omega$, where $T > 0$ and Ω is the bounded interval

$$\Omega = \{-a < x < a\},$$

with initial conditions

$$h(0,x) = h_0(x), \quad h_0 \in H^p(\Omega) \tag{68}$$

and with *no-flux* boundary conditions

$$h_x = h_{xx} = h_{xxx} = 0 \;\; for \;\; x \in \{-a, a\}. \tag{69}$$

Note that in (67) we write an alternate form of the equation (66) useful for certain calculations.

Fourth order parabolic equations do not have a maximum principle. Nonetheless, as in Bernis and Friedman's investigation of $p = 2$ case [14], we shall prove

$$\text{initial data} \geq 0 \Rightarrow \text{the solution} \geq 0.$$

Clearly this is wrong for the linear fourth order equation $h_t + h_{xxxx} = 0$. Due to the lack of maximum principle, one must rely on proving dissipation results for nonlinear entropies. Singularity formation of the form $h \to 0$ is therefore an interesting question for the fourth order nonlinear degenerate parabolic equations.

A main objective of this study is to provide a range of dissipated entropy functionals that are useful for nonnegativity, positivity and long time behavior of solutions. We divide our results into categories as follows.

I. **Nonnegativity**

Using the ideas of Bernis and Friedman [14] we seek for zero'th order Lyapunov functionals which may be useful for proving nonnegativity of solutions to the equation (67), combined with the initial and boundary conditions (68) and (69) respectively. To this end, we define

$$E_0[h(t,x)] := \int_\Omega \Phi(h(t,x))dx, \quad where \;\; \Phi''(s) = \frac{1}{s^n} \tag{70}$$

and prove that E_0 satisfies

$$E_0[h(t,x)] + (p-1)\int_0^t \int_\Omega (h_x^2)^{p/2-1} h_{xx}^2 dxdt = E_0[h_0(x)]. \tag{71}$$

Conclusion of these calculations is that

$$h(t,x) > 0 \text{ for } n \geq 2 + \tfrac{p}{p-1}.$$

II. **Regularization**

Analogous to the thin-film equation case, we define

$$P_\epsilon(h) := \frac{h^{(2+\frac{p}{(p-1)})} h^n}{\epsilon h^n + h^{(2+\frac{p}{(p-1)})}}, \tag{72}$$

and consider the equation

$$h_t = -(p-1)[P_\epsilon(h)[(h_x^2)^{p/2-1} h_x]_{xx}]_x. \tag{73}$$

The initial condition of the problem is also modified:

$$h_{0\epsilon}(x) = h_0(x) + \epsilon^\theta, \quad 0 < \theta < 2/5. \tag{74}$$

Finally, the boundary conditions (69) are kept unchanged. We prove the following theorem, which states the properties of a weak solution obtained by uniform limit as $\epsilon \to 0$ of solutions h_ϵ of the regularized problems.

Theorem 1 (Properties of positively approximated solution) : Any function h obtained by letting $\epsilon_k \to 0$ so that $h_{\epsilon_k} \to h$ in $C_{loc}(\bar{Q})$ as $k \to \infty$, where $\{h_\epsilon\}$ is a sequence of solutions to the regularized problem (73), (74) and (69) satisfies:

$$h \in C^{\beta,1/p'}_{t,x}(\bar{Q}_T), \tag{75}$$

where $Q_T := (0,T)\times\Omega$, $\frac{1}{p}+\frac{1}{p'} = 1, \beta = \frac{p-1}{5p-2}$ and $f \in C^{a,b}_{t,x}$ means that the function f is Hölder continuous of order a and b in t and x respectively.

$$h_t, h_x, h_{xx}, h_{xxx}, h_{xxxx} \in C(P), \tag{76}$$

where $P = \bar{Q}_T - (\{h = 0\} \cup \{t = 0\})$, and

$$P_\epsilon(h)[(h_x^2)^{p/2-1}h_{xx}]_x \in L^2(P), \tag{77}$$

h satisfies (73) in the following sense:

$$\iint_{Q_T} h\phi_t dxdt + (p-1)\iint_P h^n[(h_x^2)^{p/2-1}h_{xx}]_x\phi_x dxdt = 0, \tag{78}$$

for all ϕ Lipschitz in $\bar{Q}_T$, and $\phi = 0$ near $t = 0$ and near $t = T$,

$$h(0,x) = h_0(x), \quad x \in \bar{\Omega}, \tag{79}$$

$$h_x(t,\cdot) \to h_{0x} \text{ strongly in } L^p(\Omega) \text{ as } t \to 0, \tag{80}$$

and finally h satisfies the boundary conditions (69) at all points of the lateral boundary where $h \neq 0$.

This kind of regularization is also useful for improving the result of singularity formation. Indeed we deduce that for $n \geq 2 + p' - \frac{1}{p'}$, where $p' = \frac{p}{p-1}$ singularity formation[1] is not possible.

Remark : There is also another regularization which is somewhat standard in the theory of nonlinear degenerate parabolic equations. This regularization, first introduced by Bernis and Friedman in [14] for the thin-film equation, reads as follows:

$$h_{\epsilon t} = -(p-1)\left((h_\epsilon^n + \epsilon)[(h_{\epsilon x}^2 + \epsilon)^{\frac{p}{2}-1} h_{\epsilon x}]_{xx}\right)_x, \quad \epsilon > 0. \tag{81}$$

Using this kind of a regularization one can show the existence of a weak solution which is a uniform limit as $\epsilon \to 0$ of solutions to (81). The details of this regularization are provided here.

III. **Entropy dissipation-entropy estimate**

We prove that the functional $K_q(h(t,x)) := \int_\Omega \frac{h_x^2}{h^q} dx$ is an *entropy functional* for positive smooth solutions of (61) with $p = 3$ and $n = 2$. We bound the rate of decrease of K_q in terms of itself along any smooth positive solution of (61) with $p = 3$ and $n = 2$. More precisely, we prove that there exists a constant $C > 0$ such that

$$K_q(h(t,x)) \leq \left[\frac{2}{5(Ct + \frac{2}{5}[K_q(0)]^{-5/2})}\right]^{2/5}. \tag{82}$$

This clearly gives an initial polynomial decay(like $t^{-2/5}$) of positive smooth solutions to the equilibrium and once $K_q(h(t,x))$ is small enough we can then use linearization to obtain an exponential decay.

For the sake of completeness we also prove that an inequality of the form (82) is valid for positive smooth solutions of the thin film equation (41) in the physical cases $n = 1$ and $n = 2$. To illustrate how the linearization works we provide the details of the linearization in the thin film equation case (41) for $n = 1$.

IV. **Regularity and Large-time behavior**

We prove the following result related to the regularity properties of solutions.

[1]as singularity formation we mean $h \to 0$ throughout.

Theorem 2 (Regularity) :

Let $0 < n < 3$ and let h_0 satisfy (85) and let h_ϵ be the solution of the problem (73) with initial condition

$$h_{0\epsilon}(x) = h_0(x) + \delta(\epsilon),$$

and boundary conditions (69), where $P_\epsilon(s)$ is given by (72). Let h be a solution of the equation (67) with initial and boundary conditions (68) and (69) obtained by $h_{\epsilon_k} \to h$, as $\epsilon_k \to 0$.

Set

$$b_n = \begin{cases} \frac{p}{(p-1)} & \text{if } 0 < n \leq 3\frac{(p-1)}{p} \\ \frac{3}{n} & \text{if } 3\frac{(p-1)}{p} \leq n < 3, \end{cases}$$

Then, for any $b \in (0, b_n)$,

$$h^{1/b}(t, \cdot) \in C^1([-a, a]), \text{for almost every} t > 0. \tag{83}$$

Remark. Since $b_n > 1$ for $0 < n < 3$ we my substitute $b_n = 1$ in (83) and obtain that $h(t, x) \in C^1([-a, a])$ for almost every $t > 0$.

Theorem 3 (large-time behavior) : Let h and h_0 be as in Theorem 1, then we have that

$$h(t, \cdot) \to \frac{1}{2a} \int_{-a}^{a} h_0(x)dx \ \text{ uniformly in } [-a, a] \text{ as } t \to \infty. \tag{84}$$

V. Support Properties

We prove the following result related to the support properties of solutions.

Theorem 4 (Support properties) :

Let h_0 satisfy

$$n \in (0, \infty), \ \ 0 \leq h_0 \in H^p(\Omega), \ \ h_0 \not\equiv 0 \ \ in \ \ [-a, a] \tag{85}$$

and let h_ϵ be the solution of the equation (73) with initial condition

$$h_{0\epsilon}(x) = h_0(x) + \delta(\epsilon), \tag{86}$$

and boundary conditions (69), where $P_\epsilon(s)$ is given by (72). Let h be a solution of the problem (67), (68) and (69), obtained by

$$h_{\epsilon_k} \to h \ in \ C_{loc}(\bar{Q}) \ \ as \ \ \epsilon_k \to 0. \tag{87}$$

Then, one has the following conclusions.

(i) If $n \geq 1 + \frac{(p-1)}{p}$, then

$$supp(h(t_0, \cdot)) \subseteq supp(h(t, \cdot)) \text{ for } t > t_0.$$

(ii) If $n > \frac{p}{p-1}$ then

$$h(t_0, x_0) > 0 \implies h(t, x_0) > 0 \text{ for almost every } t > t_0.$$

(iii) If $n \geq 1 + \frac{(p-1)}{p} + \frac{p}{(p-1)}$, then

$$h(t_0, x_0) > 0 \implies h(t, x_0) > 0 \text{ for all } t > t_0.$$

(iv) If $0 < n < 3$ and $h_{0\epsilon}$ satisfies (168), then there exists $T = T_{h_0} \geq 0$ such that

$$h(t, x) > 0 \;\; for \;\; |x| \leq a, \;\; t > T. \tag{88}$$

(v) If $n \geq 2 + \frac{p}{(p-1)}$, then

$$supp(h(t, \cdot)) = supp(h_0), \quad for \;\; t > 0. \tag{89}$$

Remark : In order to prove (v) we need to show that for $n \geq 2 + p/(p-1)$, one has

$$supp(h(t, \cdot)) \subseteq supp(h_0), \quad t > 0,$$

by (i). We also note that (iv) follows from (84).

VI. **Asymptotic behavior of nonnegative solutions**

Using the energy functional (65) we deduce the long time behavior of both the smooth and the weak solutions. The following result, which is a generalization of the results of [93], is quite useful for this purpose.

Lemma 5 (A useful inequality) : For any measurable function $\psi : [0, \infty) \to [0, \infty)$ and for any $0 \leq u \in H^3(\Omega)$ with $u_x(\pm a) = 0$, we have that

$$\left(\int_\Omega \frac{u^2}{\psi(u)} dx\right)^{1/2} \left(\int_\Omega \psi(u)[(u_x^2)^{p/2-1} u_{xx}]_x^2 dx\right)^{1/2} \geq CH_p(u), \tag{90}$$

where C is a finite constant depending on a and p. Using this Lemma we deduce the following proposition, which is useful for obtaining long time behavior result for nonnegative smooth solutions.

Proposition 6 (Energy dissipation bound) : Suppose that $0 < n < \infty$ and h is a nonnegative smooth solution(i.e.classical solution) of the equation (67) with initial and boundary conditions (68) and (69). Moreover, suppose that the initial condition $h_0 \in H^1(\Omega)$ has finite mass. Then, we have

(i) If $0 < n < 2$, then there exists a constant $0 < C = C(\|h_{0x}\|_{L^p(\Omega)}, p, a, n)$ such that

$$\int_{\Omega} h^n(t,x)[(h_x^2)^{p/2-1}h_{xx}]_x dx \geq C[H_p(h(t,x))]^2, \forall t > 0. \tag{91}$$

(ii) If $n = 2$ then (91) holds with $C = C(p, a)$. i.e. C is now independent of $\|h_{0x}\|_{L^p(\Omega)}$.

(iii) If $n > 2$ and $\int_{\Omega} h_0^{2-n}(x)dx < \infty$, then there exists a constant $0 < C = C(\int_{\Omega} h_0^{2-n}(x)dx, p, a)$ such that (91) holds.

By the Proposition we deduce that

$$H_p[h(t,x)] \leq [H_p[h_0]^{-1} + Ct]^{-1}, t > 0. \tag{92}$$

Hence, from this, $H_p(h)$ becomes sufficiently small after some finite time and so $h(t,x)$ becomes uniformly bounded from below away from 0. From this point on we can then deduce from linearization that there is an exponential decay.

Note that we could not deduce the long time behavior of weak solutions using entropy dissipation-entropy estimate section. However, using the usual energy it is possible to prove the following result.

Proposition 7 (Energy dissipation for weak solutions) : Assume that $n \in (0,1) \cap (2,\infty)$ and $h_0 \in H^1(\Omega)$ satisfies $\int_{\Omega} h_0^{2-n} dx < \infty, n > 2$ and has finite mass. Then, there exists a constant $C = C(\int_{\Omega} h_0^{2-n} dx, p, a, n) > 0$ such that

$$\frac{dH_p[h(t,x)]}{dt} \leq -C\left(H_p[h(t,x)]\right)^2, \forall t > 0. \tag{93}$$

where $h(t,x)$ is a weak solution of the equation (67) with initial and boundary conditions (68) and (69).

Clearly the Proposition 7 yields that

$$H_p[h(t,x)] \leq H_p[h_0]\left(1+\tau_1 H_p[h_0]t\right)^{-1}, \tau_1 > 0. \tag{94}$$

This implies that whenever $H_p[h(t,x)]$ is small enough $h(t,x)$ becomes bounded below away from 0, and after this point on we have exponential decay by linearization.

For the remaining case for n we need more work to deduce the long time behavior of weak solutions. Here we prove the exponential decay directly. The details are given below.

Proposition 8 (Exponential decay of weak solutions for $0 < n \leq 2$**) :** Assume $0 < n \leq 2$ and $h_0 \in H^1(\Omega)$ is such that $H_p[h_0(x)] \leq K < \infty$. Then, there exists a constant $C > 0$ depending only on $H_p[h_0]$ and n such that the weak solution h of the equation (67) with initial and boundary conditions (68) and (69) satisfies

$$H_p[h(t,x)] \leq H_p[h_0]\exp(-Ct), \forall t > 0.$$

2.3 Results on the Cauchy Problem (95)

At the end of the thesis we investigate the large–time behavior of solutions to the thin film type Cauchy problem (95) given by

$$\begin{aligned} h_t + (hh_{xxx})_x &= 0, \qquad x \in \mathbb{R}, t > 0 \\ h(0,x) &= h_0(x), \qquad x \in \mathbb{R}. \end{aligned} \tag{95}$$

It was shown in previous work of Carrillo and Toscani [27] that for non negative initial data h_0 that belongs to $H^1(\mathbb{R})$ and also has a finite mass and second moment, the solutions relax in the $L^1(\mathbb{R})$ norm at an explicit rate to the unique self–similar source type solution with the same mass. The equation itself is gradient flow for an energy functional that controls the $H^1(\mathbb{R})$ norm, and so it is natural to expect that one should also have convergence in this norm. Carrillo and Toscani raised this question [27], but their methods, using a different Lyapunov functions that arises in the theory of the porous medium equation, do not directly address this: Their Lyapunov functional does not involve derivatives of h. Here we show that the solutions do indeed converge in the $H^1(\mathbb{R})$ norm at an explicit, albeit slow rate, though we require more than a second moment: we present the argument assuming a fourth moment. The key to establishing convergence is that this is an *asymptotic equipartition of the excess energy* part. The energy functional whose dissipation drives the evolution through gradient flow consists of two parts: one involving derivatives of h, and one that does not. We show that these must decay at related rates – due to the asymptotic equipartition – and then use the results of Carrillo and Toscani [27] to control the rate for the part that does not depend on derivatives. From this, one gets a rate on the dissipation for all of the excess energy.

CHAPTER III

DETAILS ON A NEW FAMILY OF PARABOLIC EQUATIONS

3.1 Introduction

We now provide the details of our analysis on a new family of degenerate parabolic equations (67), derived in the previous chapter. Some of our results here also includes the thin film equation, which is a special case of (67).

In order to prove the existence of a solution to (67) we follow the methods of [14]. We consider the equation (67) in $Q_T := (0,T)\times\Omega$, where $T > 0$ and Ω is the bounded interval

$$\Omega = \{-a < x < a\}$$

with initial conditions (68) and with *no-flux* boundary conditions (69). We note that (67) is degenerate at $h = 0$, and so we approximate it by a family of non degenerate parabolic equations of the form

$$h_{\epsilon t} = -(p-1)\left((h_\epsilon^n+\epsilon)[(h_{\epsilon x}^2+\epsilon)^{\frac{p}{2}-1}h_{\epsilon x}]_{xx}\right)_x, \quad \epsilon > 0. \tag{96}$$

We also note that (96) is derived by variational consideration of the following "approximate" interface Hamiltonian

$$H_p^\epsilon := \frac{1}{p}\int_\Omega (h_x^2+\epsilon)^{\frac{p}{2}}dx. \tag{97}$$

The initial condition h_0 is approximated in $H^p(\Omega)-$norm by the functions $h_{0\epsilon} \in C^{4+\alpha}, \alpha \in (0,1)$. We note that $h_{0\epsilon}$ satisfies (69) and (68) is replaced by

$$h_\epsilon(0,x) = h_{0\epsilon}(x). \tag{98}$$

By employing the parabolic Schauder estimates [14],[39],[40] we can deduce that the initial boundary value problem (96), (69) and (98) has a unique solution for sufficiently small time interval. Let $Q_{T_0} = (0,T_0)\times\Omega$ be the set in which this unique solution exists. The derivatives

$$h_{\epsilon t}, h_{\epsilon x}, h_{\epsilon xx}, h_{\epsilon xxx}, h_{\epsilon xxxx}$$

are all Hölder continuous in $\bar{Q}_{T_0}$. Below we prove that the solution h_ϵ of (96), (69) and (98) satisfies a priori Hölder estimate in every domain Q_{T_0} arbitrary of T_0. Then we can extend solution step by step to all of Q_T.

Formally differentiate the "approximate" interface Hamiltonian (97) along the solution h_ϵ of (96). Then, it is not difficult to obtain that

$$\begin{aligned}\frac{1}{p}\int (h_{\epsilon x}^2+\epsilon)^{\frac{p}{2}}|_{t=T}dx \quad &+ \quad (p-1)\iint (h_\epsilon^n+\epsilon)[h_{\epsilon x}(h_{\epsilon x}^2+\epsilon)^{\frac{p}{2}-1}]_{xx}^2 dxdt \\ &= \quad \frac{1}{p}\int (h_{\epsilon x}^2+\epsilon)^{\frac{p}{2}}|_{t=0}dx. \end{aligned} \tag{99}$$

Note that (99) implies

$$\int_\Omega (h_{\epsilon x}^2(t,x)+\epsilon)^{\frac{p}{2}}dx \le \int_\Omega (h_{0\epsilon,x}^2(x)+\epsilon)^{\frac{p}{2}}dx. \tag{100}$$

Now, for fixed $t\in(0,T)$, one has that

$$\begin{aligned}|h_\epsilon(t,x)-h_\epsilon(t,y)| \quad &\le \quad \int_x^y |h_{\epsilon x}(z)|dz \le \int_x^y (h_{\epsilon x}^2(z)+\epsilon)^{1/2}dz \\ &\le \quad (\int_x^y (h_{\epsilon x}^2+\epsilon)^{p/2}dz)^{1/p}|x-y|^{1/p'},\end{aligned} \tag{101}$$

where $p'=\frac{p}{p-1}$. Integrating the equation (96) over $(0,T)\times\Omega$ one also has that

$$\int_\Omega h_{\epsilon x}(T,x)dx = \int_\Omega h_{0,x}(x)dx. \tag{102}$$

Also, by the approximation procedure of the initial data we have,

$$\int_\Omega (h_{0\epsilon,x}^2+\epsilon)^{p/2} \le (1+a(\epsilon))\int_\Omega (h_{0,x}^2+\epsilon)^{p/2}, \tag{103}$$

where $a(\epsilon)\to 0$ as $\epsilon\to 0$. From (244) and (102) we deduce that

$$|h_\epsilon(t,x)|\le K \;\; in \;\; Q_{T_0}, \tag{104}$$

and here K is a constant independent of ϵ and T_0.

Moreover, from (101) and (103) we also obtain that

$$|h_\epsilon(t,x)-h_\epsilon(t,y)|\le C|x-y|^{1/p'}, \tag{105}$$

where $p' = \frac{p}{p-1}$. We also conclude in the light of these estimates that

$$(h_\epsilon^n + \epsilon)[h_{\epsilon x}(h_{\epsilon x}^2 + \epsilon)^{\frac{p}{2}-1}]_{xx} \in L^2(Q_{T_0})$$

and moreover there exists a constant K_1 independent of T_0 and ϵ such that

$$\iint_{Q_{T_0}} (h_\epsilon^n + \epsilon)[h_{\epsilon x}(h_{\epsilon x}^2 + \epsilon)^{\frac{p}{2}-1}]_{xx}^2 dxdt \le K_1. \tag{106}$$

Lemma 9 (Hölder continuity in t) : There exists a constant M independent of T_0 and ϵ such that

$$|h_\epsilon(t_1, x) - h_\epsilon(t_2, x)| \le M|t_1 - t_2|^\beta \tag{107}$$

for all $x \in \Omega$ and $t_1, t_2 \in (0, T_0)$, and $\beta = \frac{p-1}{5p-2}$.

Proof: Suppose on the contrary that

$$|h_\epsilon(t_2, x_0) - h_\epsilon(t_1, x_0)| > M|t_2 - t_1|^\beta, \tag{108}$$

for some x_0, t_1 and t_2. To complete the proof we will derive an upper bound on M that is independent of T_0 and ϵ. Without loss of generality we suppose that $h_\epsilon(t_2, x_0) > h_\epsilon(t_1, x_0)$ and $t_2 > t_1$. Hence, (108) becomes

$$h_\epsilon(t_2, x_0) - h_\epsilon(t_1, x_0) > M(t_2 - t_1)^\beta, \tag{109}$$

where $0 < t_1 < t_2 < T_0$, and $\beta = \frac{p-1}{5p-2}$. Notice that h_ϵ satisfies

$$\iint_{Q_{T_0}} h_\epsilon \phi_t dxdt = -\iint_{Q_{T_0}} z_\epsilon \phi_x dxdt, \tag{110}$$

where $z_\epsilon = (h_\epsilon^n + \epsilon)[(h_{\epsilon x}^2 + \epsilon)^{p/2-1} h_{\epsilon xx}]_x$ and ϕ is a reasonable test function.

Since $h_{\epsilon t}$ is continuous in $\bar{Q}_{T_0}$ and $z_\epsilon = 0$ on the lateral boundary, we take ϕ as $\phi \in Lip(Q_{T_0})$, $\phi = 0$ near $t = 0$ and near $t = T_0$. Note also that ϕ is not necessarily 0 on the lateral boundary. We define ϕ by

$$\phi(t, x) = \psi(x)\theta_\delta(t),$$

where the functions ψ and θ will be defined below.

Definition of ψ : Recall the following relations for h_ϵ,

$$|h_\epsilon(t,x_2)-h_\epsilon(t,x_1)| \le K|x_2-x_1|^{\frac{p-1}{p}} \tag{111}$$

and $|z_\epsilon|_{L^2(Q_{T_0})} \le A$, where A is independent of ϵ and T_0. Define

$$\psi(x) := \psi_0\left(\frac{x-x_0}{\frac{M^{p/(p-1)}}{4^{p/(p-1)}K^{p/(p-1)}}(t_2-t_1)^{p/(p-1)\beta}}\right), \tag{112}$$

where $\beta = \frac{p-1}{5p-2}$. Here M is from (108) and K is from (111). Moreover, the function $\psi_0(x)$ satisfies $\psi_0(x)=\psi_0(-x), \psi_0 \in C_0^\infty, \psi_0(x)=1, 0\le x\le \frac{1}{2}, \psi_0(x)=0$ if $x \ge 1$, $\psi_0'(x) \le 0$ if $x \ge 0$. Notice that we have

$$\psi(x) = \begin{cases} 0 & \text{if } |x-x_0| \ge \frac{M^{p/(p-1)}}{4^{p/(p-1)}K^{p/(p-1)}}(t_2-t_1)^{p/(p-1)\beta} \\ 1 & \text{if } |x-x_0| \le \frac{1}{2}\frac{M^{p/(p-1)}}{4^{p/(p-1)}K^{p/(p-1)}}(t_2-t_1)^{p/(p-1)\beta} \end{cases}$$

Definition of θ_δ : Define

$$\theta_\delta(t) = \int_{-\infty}^{t} \theta_\delta'(s)ds,$$

where

$$\theta_\delta'(t) = \begin{cases} \frac{1}{\delta} & \text{if } |t-t_2| < \delta \\ -\frac{1}{\delta} & \text{if } |t-t_1| < \delta \\ 0 & \text{elsewhere,} \end{cases}$$

and $\delta < \frac{1}{2}(t_2-t_1)$. Note that θ_δ is Lipschitz continuous and that $|\delta_\theta| \le 1, \theta_\delta = 0$ near $t=0$ and near $t=T_0$, if δ is small enough. Plugging this function $\phi(t,x)$ into (110) yields that

$$\iint_{Q_{T_0}} h_\epsilon\psi(x)\theta_\delta'(t)dxdt = -\iint_{Q_{T_0}} z_\epsilon\psi'(x)\theta_\delta(t)dxdt. \tag{113}$$

As $\delta \to 0$ the left hand side of this equality satisfies

$$\iint_{Q_{T_0}} h_\epsilon\psi(x)\theta_\delta(t)dxdt \to \int_\Omega \psi(x)(h_\epsilon(t_2,x)-h_\epsilon(t_1,x))dx, \quad as \;\; \delta\to 0. \tag{114}$$

We will find a lower bound for this last expression. By the definition of $\psi(x)$ we only need to consider

$$|x-x_0| \le \frac{M^{p/(p-1)}}{4^{p/(p-1)}K^{p/(p-1)}}(t_2-t_1)^{p/(p-1)\beta}.$$

For such values of x, we obtain that

$$\begin{aligned} h_\epsilon(t_2,x)-h_\epsilon(t_1,x) &= [h_\epsilon(t_2,x)-h_\epsilon(t_2,x_0)] \\ &+ [h_\epsilon(t_2,x_0)-h_\epsilon(t_1,x_0)]+[h_\epsilon(t_1,x_0)-h_\epsilon(t_1,x)] \\ &\geq -2K|x-x_0|^{(p-1)/p}+M(t_2-t_1) \\ &\geq \frac{M}{2}(t_2-t_1)^\beta. \end{aligned} \tag{115}$$

Assume without loss of generality that the set $\{\psi=1\}$ is included in Ω. Then,

$$\int_\Omega \psi(x)(h_\epsilon(t_2,x)-h_\epsilon(t_1,x))dx \geq \frac{M}{2}(t_2-t_1)^\beta \frac{M^{p/(p-1)}}{4^{p/(p-1)}K^{p/(p-1)}}(t_2-t_1)^{p/(p-1)\beta}. \tag{116}$$

On the other hand, the right hand side of (113) can be bounded from above by means of the Hölder's inequality and definitions of ψ and θ_δ. Indeed, we have

$$|\iint_{Q_{T_0}} z_\epsilon \psi'(x)\theta_\delta(t)dxdt| \leq \left(\iint_{Q_{T_0}} z_\epsilon^2 dxdt\right)^{1/2}\left(\iint_{Q_{T_0}}[\psi'(x)\theta_\delta(t)]^2 dxdt\right)^{1/2}$$

$$\leq \frac{C_1}{\frac{M^{p/(p-1)}}{4^{p/(p-1)}K^{p/(p-1)}}(t_2-t_1)^{p/(p-1)\beta}}\left(\iint_{Q_{T_0}} z_\epsilon^2 dxdt\right)^{1/2}\frac{\sqrt{2}}{4^{p/(2(p-1))}}\frac{M^{p/2(p-1)}}{K^{p/2(p-1)}}(t_2-t_1)^{\frac{p}{2(p-1)}\beta}(t_2-t_1+\delta)^{1/2}.$$

Therefore, by letting $\delta\to 0$, we deduce that

$$M^{(\frac{p}{p-1}+1)}(t_2-t_1)^{(\frac{p}{p-1}+1)\beta} \leq \frac{C_2}{M^{p/2(p-1)}}(t_2-t_1)^{1/2-(\frac{p}{p-1}+1)\beta}, \tag{117}$$

where C_2 is a constant independent of ϵ, M and T_0. Since $\beta=\frac{p-1}{5p-2}$ we have $M\leq C_2^{\frac{2(p-1)}{5p-2}}$. Thus, the proof of the lemma is complete.

$\square$

From these calculations we deduce that there is an upper bound on the $C_{t,x}^{\frac{p-1}{5p-2},(p-1)/p}$−norm of h_ϵ in Q_{T_0}, which is independent of ϵ and T_0. This allows us to extend h_ϵ step by step to all of Q_T. Moreover, we deduce also that the sequence $\{h_\epsilon\}$ is uniformly bounded and equi-continuous family in $\bar{Q}_T$, and therefore by the Arzelá-Ascoli theorem, for every sequence $\epsilon\to 0$ there is a subsequence $\{\epsilon_k\}$ such that

$$h_{\epsilon_k}\to h \ \text{ uniformly } \text{ in } \bar{Q}_T. \tag{118}$$

Hence h has the same continuity properties as h_ϵ.

3.2 Existence of Weak Solution

We consider the problem (67) combined with the initial and boundary conditions (68) and (69). The following theorem establishes the existence of a weak solution and also provides some properties of this weak solution.

Definition (Weak solution) : The function h satisfying the assertions of the following theorem is called a *weak solution* of the problem (67) combined with the initial and boundary conditions (68) and (69).

Theorem 10 (Properties of a weak solution) : Any function obtained by (118) satisfies:

$$h \in C(\bar{Q}_T), \quad \text{actually} \quad h \in C_{t,x}^{\beta,1/p'}(\bar{Q}_T) \tag{119}$$

where $p' = \frac{p}{p-1}$ and $\beta = \frac{p-1}{5p-2}$,

$$h_t, h_x, h_{xx}, h_{xxx}, h_{xxxx} \in C(P), \tag{120}$$

where $P = \bar{Q}_T - (\{h = 0\} \cap \{t = 0\})$, and

$$h^n[(h_x^2)^{p/2-1}h_{xx}]_x \in L^2(P), \tag{121}$$

h satisfies (67) in the following sense:

$$\iint_{Q_T} h\phi_t dxdt + (p-1)\iint_P h^n[(h_x^2)^{p/2-1}h_{xx}]_x\phi_x dxdt = 0, \tag{122}$$

for all ϕ that is Lipschitz in $\bar{Q}_T$, and $\phi = 0$ near $t = 0$ and near $t = T$,

$$h(0, x) = h_0(x), \quad x \in \bar{\Omega}, \tag{123}$$

$$h_x(t, \cdot) \to h_{0x} \;\; \text{strongly in } L^p(\Omega) \text{ as } t \to 0, \tag{124}$$

and finally h satisfies the boundary conditions (69) at all points of the lateral boundary where $h \neq 0$.

Proof : We obviously have

$$h(0, x) = h_0(x),$$

and

$$h(t,x) \in C^{\beta,1/(p')}_{t,x}.$$

Let ϕ be as indicated. Then we have that

$$\begin{aligned}\iint_{Q_T} h_\epsilon \phi_t dxdt \quad &+ \quad (p-1)\iint_{Q_T} h_\epsilon^n [(h_{\epsilon x}^2)^{p/2-1} h_{\epsilon xx}]_x \phi_x dxdt \\ &+ \quad (p-1)\epsilon \iint_{Q_T} [(h_{\epsilon x}^2)^{p/2-1} h_{\epsilon xx}]_x \phi_x dxdt = 0. \qquad (125)\end{aligned}$$

We know that

$$\epsilon(p-1)\iint_{Q_T} [(h_{\epsilon x}^2)^{p/2-1} h_{\epsilon xx}]_x^2 dxdt \le C,$$

and by Cauchy-Schwartz inequality we deduce that

$$\begin{aligned}\epsilon(p-1)\iint_{Q_T} [(h_{\epsilon x}^2)^{p/2-1} h_{\epsilon xx}]_x \phi_x dxdt \quad &\le \quad C(p-1)\left(\iint_{Q_T} [(h_{\epsilon x}^2)^{p/2-1} h_{\epsilon xx}]_x^2 dxdt\right)^{1/2}\left(\iint_{Q_T} \phi_x^2 dxdt\right)^{1/2} \\ &\le \quad \epsilon C(p-1) \to 0, \qquad (126)\end{aligned}$$

as $\epsilon \to 0$ since C is a constant independent of ϵ, and $p > 2$. Hence, we conclude that

$$\epsilon(p-1)\iint_{Q_T} [(h_{\epsilon x}^2)^{p/2-1} h_{\epsilon xx}]_x \phi_x dxdt \to 0, \quad as\ \epsilon \to 0. \qquad (127)$$

Let $z_\epsilon := (h_\epsilon^n + \epsilon)[(h_{\epsilon x}^2)^{p/2-1} h_{\epsilon xx}]_x$. We have from results of the previous section that

$$|z_\epsilon|_{L^2(Q_T)} \le A,$$

where A is a constant independent of ϵ and T. We deduce that z_ϵ has a subsequence such that $z_\epsilon \to z$ weakly in $L^2(Q_T)$.

By regularity theory of uniformly parabolic equations and the uniform Hölder continuity of h_ϵ we deduce that

$$h_{\epsilon t}, h_{\epsilon x}, h_{\epsilon xx}, h_{\epsilon xxx}, h_{\epsilon xxxx}$$

are uniformly convergent in any compact subset of P. Hence,

$$z = h^n [(h_x^2)^{p/2-1} h_{xx}]_x.$$

h satisfies the no-flux boundary conditions (69) at all points of the lateral boundary where $h \neq 0$.

$$z_\epsilon \to z = h^n [(h_x^2)^{p/2-1} h_{xx}]_x,$$

weakly in $L^2(Q_T)$. This implies that

$$h^n[(h_x^2)^{p/2-1}h_{xx}]_x \in L^2(Q_T).$$

Let $\delta > 0$ be arbitrary. By these observations we have that

$$(p-1)\iint_{\{|h|>\delta\}} h_\epsilon^n[(h_{\epsilon x}^2)^{p/2-1}h_{\epsilon xx}]_x\phi_x dxdt \to (p-1)\iint_{\{|h|>\delta\}} h^n[(h_x^2)^{p/2-1}h_{xx}]_x\phi_x dxdt.$$

Now, take ϵ sufficiently small, depending on δ, and consider

$$\begin{aligned}
&(p-1)|\iint_{\{|h|\le\delta\}} h_\epsilon^n[(h_{\epsilon x}^2)^{p/2-1}h_{\epsilon xx}]_x\phi_x dxdt| \\
&\le (p-1)\left(\iint_{\{|h|\le\delta\}} h_\epsilon^n[(h_{\epsilon x}^2)^{p/2-1}h_{\epsilon xx}]_x^2 dxdt\right)^{1/2}\left(\iint_{\{|h|\le\delta\}} h_\epsilon^n\phi_x^2 dxdt\right)^{1/2} \\
&\le C(p-1)\delta^n,
\end{aligned} \tag{128}$$

where we have used the dissipation result (99). Note that $h_{0\epsilon} \to h_0$ in $H^p(\Omega)$ and also

$$\int_\Omega |h_{\epsilon x}(t,x)|^p dx \le \int_\Omega |h_{0\epsilon x}(x)|^p dx.$$

Hence, we deduce that

$$\lim sup_{t\to 0}\int_\Omega |h_x(t,x)|^p dx \le \int_\Omega |h_{0x}|^p dx.$$

Note also that $h_x(t,\cdot) \to h_{0x}$ weakly in $L^p(\Omega)$. Thus, letting $\epsilon \to 0$ and noting that $\delta > 0$ is arbitrary we conclude the proof.

$\square$

3.3 Nonnegativity and Positivity of Solutions

Let E_0 be defined by (70). Multiplying the equation (67) formally by $\Phi'(h(t,x))$ and integrating we have that

$$\frac{d}{dt}E_0[h(t,x)] = (p-1)(p-2)\iint_{Q_T}(h_x^2)^{\frac{p}{2}-1}h_{xx}^2 dxdt + (p-1)\iint_{Q_T}(h_x^2)^{\frac{p}{2}-1}h_x h_{xxx}dxdt. \tag{129}$$

Here we have used the fact that $\Phi''(s) = \frac{1}{s^n}$. From (129) one deduces that (71) is satisfied and this clearly implies that E_0 dissipates whenever $p \ge 1$. Thus, by using the Hölder continuity of $h(t,x)$ in x, we deduce that for $n \ge 2 + \frac{p}{(p-1)}$ there can not be singularity formation. Note that this was proved by Bernis and Friedman in [14] for the thin film equation (41), which is $p = 2$ case of (67).

3.4 Approximation by positive h_ϵ

The calculations above suggest the following regularization of the problem. Let $P_\epsilon(h)$ be given by (72) and consider the regularized problem (73). The initial condition of the problem is also modified, indeed we define $h_{0\epsilon}(x)$ by (74).

Since $\lim_{s\to 0} \frac{P_\epsilon(s)}{s^{2+\frac{p}{(p-1)}}} = \frac{1}{\epsilon}$ if $1 \le n < 2 + \frac{p}{(p-1)}$ and if $n \ge 2 + \frac{p}{(p-1)}$ then $\lim_{s\to 0} \frac{P_\epsilon(s)}{s^{2+\frac{p}{(p-1)}}} = 0$, and $h_{0\epsilon} > 0$, there exists a unique, positive smooth solution h_ϵ of the problem (73) combined with the initial condition (74) and with *no-flux* boundary conditions (69). We can modify the calculations done in [94] to deduce that there exists a limit function h such that $h_\epsilon \to h$ uniformly. Moreover, we will show that this limit function is a weak solution of the problem (67), (68) and (69).

Proof of Theorem 1: Take ϕ as indicated. One can easily obtain that

$$\|P_\epsilon(h)[(h_x^2)^{p/2-1}h_{xx}]_x\|_{L^2(Q_T)} \le A,$$

where A is independent of ϵ and T. Let $Z_\epsilon := P_\epsilon(h)[(h_x^2)^{p/2-1}h_{xx}]_x$. Then, for a subsequence such that $Z_\epsilon \to Z$ weakly in $L^2(Q_T)$. By regularity theory of uniformly parabolic equations and uniform Hölder continuity of h_ϵ we deduce that

$$h_{\epsilon t}, h_{\epsilon x}, h_{\epsilon xx}, h_{\epsilon xxx}, h_{\epsilon xxxx}$$

are uniformly convergent in any compact subset of $P = \bar{Q}_T - (\{h = 0\} \cup \{t = 0\})$. Hence, $h^n[(h_x^2)^{p/2-1}h_{xx}]_x = Z$, on P. Hence,

$$h_t, h_x, h_{xx}, h_{xxx}, h_{xxxx} \in C(P),$$

and

$$h^n[(h_x^2)^{p/2-1}h_{xx}]_x \in L^2(P).$$

Hence, we also see that h solves the problem (67) in the *weak* sense. For any $\delta > 0$ one has that

$$(p-1)\iint_{\{|h|>\delta\}} P_\epsilon(h_\epsilon)[(h_{\epsilon x}^2)^{p/2-1}h_{\epsilon xx}]_x\phi_x dxdt \to (p-1)\iint_{\{|h|>\delta\}} h^n[(h_x^2)^{p/2-1}h_{xx}]_x\phi_x dxdt.$$

On the other hand, if ϵ is small enough(depending on δ), then by Cauchy-Schwartz inequality,

$$|(p-1)\iint_{\{|h|\le\delta\}} P_\epsilon(h_\epsilon)[(h_{\epsilon x}^2)^{p/2-1}h_{\epsilon xx}]_x\phi_x dxdt| \le C|(p-1)|\delta^{n/2} \to 0 \ \ as \ \ \delta \to 0. \tag{130}$$

Recall also that we have

$$\int_\Omega |h_{\epsilon x}|^p(t,x)dz \le \int_\Omega |h_{0\epsilon x}|^p dx,$$

and $h_{\epsilon 0} \to h_0$ in $H^p(\Omega)$. Combining these we deduce that

$$\lim sup_{t\to 0}\int_\Omega |h_x|^p(t,x)dx \le \int_\Omega |h_{0x}(x)|^p dx,$$

and

$$h_x(t,\cdot) \to h_{0x} \ \text{ weakly } in \ L^p(\Omega) \text{ as } t \to 0.$$

We see that a weakly convergent sequence which is bounded. In fact, in this case the sequence converges strongly to the same limit. Taking $\epsilon \to 0$, since $\delta > 0$ is arbitrary, we conclude that (78) is satisfied. This completes the proof of the theorem. □

We now use this regularization scheme to improve the result of singularity formation. To this end, define the natural entropy by

$$H_\epsilon(h) := \int_\Omega G_\epsilon(h(t,x))dx, \tag{131}$$

where G_ϵ satisfies

$$G_\epsilon(h(t,x))'' P_\epsilon(h(t,x)) = h^\beta(t,x), \ \ \beta < 0. \tag{132}$$

The more negative β is, the better result we obtain for the singularity formation. We can easily determine $G_\epsilon(h(t,x))$ using (132), where we choose the constant of the integration so that $\int_\Omega G_\epsilon(h(t,x))dx \ge 0$. Multiplying the equation (67) formally by $G'(h(t,x))$ and applying integration by parts we obtain that

$$\begin{aligned}\int_\Omega G_\epsilon'(h)h_t dx &= -(p-1)^2\int_\Omega h^\beta(h_x^2)^{p/2-1}h_{xx}^2 dx + \frac{\beta(\beta-1)(p-1)^2}{(p+1)}\int_\Omega h^{\beta-2}(h_x^2)^{p/2-1}h_x^4 dx.\\ &=: -c_1J_1(h(t,x)) + c_2J_2(h(t,x)).\end{aligned} \tag{133}$$

To proceed further we need the following lemma.

Lemma 11 (Negative term beats in (133)): One has the following inequality for $0 \leq h \in H^3(\Omega)$, satisfying $h_x(\pm a) = 0$,

$$J_1(h(t,x)) \geq C J_2(h(t,x)), \tag{134}$$

where

$$C := \frac{(1-\beta)^2}{(p+1)^2}.$$

Proof: Inequality (134) can be verified easily by considering that, for any constant $A > 0$,

$$0 \leq \int_\Omega \left[h^{\beta/2}((h_x^2)^{p/2-1})^{1/2} h_{xx} - A h^{\beta/2-1}((h_x^2)^{p/2-1})^{1/2} h_x^2 \right]^2 dx. \tag{135}$$

By employing integration by parts we see that (135) is equivalent to

$$J_1(h(t,x)) + (A^2 - \frac{2A(1-\beta)}{(p+1)}) J_2(h(t,x)) \geq 0. \tag{136}$$

Optimizing over A, in (136) we obtain (134).

□

Using (134) we finally have that

$$\frac{d}{dt} H_\epsilon(h(t,x)) \leq C_\beta J_2(h(t,x)), \tag{137}$$

where

$$C_\beta := -\frac{(p-1)^2(1-\beta)^2}{(p+1)^2} + \frac{\beta(\beta-1)(p-1)^2}{(p+1)}.$$

Note that C_β is non positive if and only if $\beta \geq -\frac{1}{p}$. Using the definition of G_ϵ and the Hölder continuity of $h(t,x)$ in x we deduce that there is no singularity formation of the form $h \to 0$ if

$$n \geq 2 + p' - \frac{1}{p},$$

where $p' = \frac{p}{p-1}$. Note that this obeys to $p = 2$ case where $n \geq 7/2$ implies no singularity formation in this case. Moreover, when $p = 3$ this shows that $n \geq 3.1666...$ implies no singularity formation in this case.

3.5 An Entropy dissipation-Entropy estimate for (66)

The term "entropy" is frequently used for a Lyapunov functional whose rate of decrease can be bounded in terms of itself. That is if $H(f)$ is some functional of f, and along the flow of some evolution we have

$$\frac{d}{dt}H(f) \leq -\Phi(H(f)), \tag{138}$$

with Φ some continuous strictly monotone increasing function on $\mathbb{R}_+$, then the functional $H(f)$ is called an entropy, and the inequality (138) is called an entropy dissipation-entropy inequality. The point is that (138) can be used to quantitatively estimate the rate of decay of $H(f)$.

Consider again a smooth solution $h(t, x)$ of (67) and define the functional K_q by,

$$K_q(h(t, x)) := \int_\Omega \frac{h_x^2}{h^q} dx. \tag{139}$$

We note that this functional has been discovered by Laugesen [61] for the thin-film equation. Laugesen showed that K_q is a Lyapunov functional for the thin-film equation provided that $q \in [0, 1/2]$. Moreover, K_q was used in [25] to prove an entropy dissipation-entropy estimate for a thin film type equation. Moreover, the special case $n = 2$ and $p = 3$ has been considered in [95], where it was noted that the same kind of calculations work for wider range of p and n values.

Differentiating K_q along a smooth positive solution of (67) yields that(integrals below are over the set Ω)

$$\begin{aligned} \frac{dK_q(h)}{dt} &= -2\int \frac{h_x}{h^q}[h^n((p-1)(p-2)(h_x^2)^{\frac{p}{2}-2}h_x h_{xx}^2 + (p-1)(h_x^2)^{\frac{p}{2}-1}h_{xxx})]_{xx}dx \\ &+ q\int \frac{h_x^2}{h^{q+1}}[h^n((p-1)(p-2)(h_x^2)^{\frac{p}{2}-2}h_x h_{xx}^2 + (p-1)(h_x^2)^{\frac{p}{2}-1}h_{xxx})]_x dx. \end{aligned} \tag{140}$$

We apply integration by parts twice to the first term and once for the second term; so that h_{xxx} is the highest order derivative appearing in the calculations. Collecting the likely terms after integrating by parts by help of some algebra, we finally get that

$$
\begin{aligned}
\frac{dK_q(h)}{dt} &= \frac{2}{3}(p-1)(p-2)(p-3)\int \frac{(h_x^2)^{\frac{p}{2}-2}h_{xx}^4}{h^{q-n}}dx \\
&+ [\frac{2}{3}(p-2)(5q+n)(q-n+1)-q(q+1)(p-1)(p-2)]\int \frac{(h_x^2)^{\frac{p}{2}-1}h_x^2h_{xx}^2}{h^{q-n+2}}dx \\
&+ 4[q(p-1)-\frac{1}{3}(p-2)(5q+n)]\int \frac{(h_x^2)^{\frac{p}{2}-1}h_xh_{xx}h_{xxx}}{h^{q-n+1}}dx \\
&- 2(p-1)\int \frac{(h_x^2)^{\frac{p}{2}-1}h_{xxx}^2}{h^{q-n}}dx - q(q+1)(p-1)\int \frac{(h_x^2)^{\frac{p}{2}-1}h_x^3h_{xxx}}{h^{q-n+2}}dx. \qquad (141)
\end{aligned}
$$

Denoting the first term by T_1, we note that using the notation in (147) and (148) below, and also defining the constants accordingly, we can rewrite (141) as

$$
\frac{dK_q(h)}{dt} = c_0T_1 + c_1I_2 + c_2J_{12} + c_3J_{13} + c_4I_1. \qquad (142)
$$

We focus on the case $p \geq 2$ in this paper, which is realistic as the first non constant term appearing in the Taylor polynomial approximation for $\sqrt{1+x^2}$ is $\frac{1}{2}x^2$. Note that for $2 \leq p \leq 3$ the first term, T_1, in (141) is non positive so that it can be neglected in the procedure. In the case $p > 3$, the first term becomes nonnegative and it does not appear in (147) or (148). Thus, to proceed further in this case one needs to bound this term in terms of the integrals in the lists (147) and (148). For the moment such a bound is not available to us, so we do not consider these cases here.

As mentioned above in the case $2 \leq p \leq 3$(note that we consider integer values only, that is we consider $p = 2$ and $p = 3$ in this case) things are relatively easier. In this case we have, by neglecting the first term, that

$$
\frac{d}{dt}K_q(h) \leq c_1I_2 + c_2J_{12} + c_3J_{13} + c_4I_1, \qquad (143)
$$

where the constants $c_i, i = 1,2,3,4$ are defined in (141) and I_1, J_{12}, J_{13} and I_2 are given in (147) and (148).

Step 1 : *We show that*

$$
\frac{dK_q(h)}{dt} \leq -C_{pqn}I_3, \qquad (144)
$$

where C_{pqn} is a positive constant which depends on p, q and n, and I_3 is given in (147).

Proof of Step 1 :

To show that the right hand side of (141) is negative, we will try to write it as a sum of negative squares. To do this, define the nonnegative quantity A by,

$$A := \int \left[\alpha h_{xxx} + \beta \frac{h_x h_{xx}}{h} + \gamma \frac{h_x^3}{h^2} \right]^2 (h_x^2)^{\frac{p}{2}-1} h^{n-q} dx, \tag{145}$$

where the numbers α, β and γ will be chosen below. (145) can be written as

$$A = \alpha^2 I_1 + \beta^2 I_2 + \gamma^2 I_3 + 2\alpha\beta J_{12} + 2\alpha\gamma J_{13} + 2\beta\gamma J_{13}, \tag{146}$$

where

$$I_1 = \int (h_x^2)^{p/2-1} \frac{h_{xxx}^2}{h^{q-n}} \mathrm{d}x, \qquad I_2 = \int (h_x^2)^{p/2-1} \frac{h_x^2 h_{xx}^2}{h^{q-n+2}} \mathrm{d}x, \qquad I_3 = \int (h_x^2)^{p/2-1} \frac{h_x^6}{h^{q-n+4}} \mathrm{d}x; \tag{147}$$

$$J_{12} = \int (h_x^2)^{p/2-1} \frac{h_x h_{xx} h_{xxx}}{h^{q-n+1}} \mathrm{d}x, \qquad J_{13} = \int (h_x^2)^{p/2-1} \frac{h_x^3 h_{xxx}}{h^{q-n+2}} \mathrm{d}x, \qquad J_{23} = \int (h_x^2)^{p/2-1} \frac{h_x^4 h_{xx}}{h^{q-n+3}} \mathrm{d}x. \tag{148}$$

Lemma 12 : Integration by parts yields the following relations:

$$I_2 = -\left(\frac{1}{(p+1)} \right) J_{13} + \left(\frac{q-n+2}{(p+1)} \right) J_{23} \tag{149}$$

$$J_{23} = \left(\frac{(q-n+3)}{p+3} \right) I_3. \tag{150}$$

Proof: This is straightforward computation. □

Since there are no useful integration by parts identities for I_1 and J_{12}, we use the definition of A appropriately to eliminate these terms from the game.

$$-\alpha^2 I_1 - 2\alpha\beta J_{12} = -A + \beta^2 I_2 + \gamma^2 I_3 + 2\alpha\gamma J_{13} + 2\beta\gamma J_{23}. \tag{151}$$

We use (151) in (142) appropriately.

For $p = 2$ **or** $p = 3$ **:** In this case we have to choose

$$\alpha := \sqrt{-c_4}, \qquad \beta = -\frac{c_2}{2\sqrt{-c_4}}. \tag{152}$$

Using (269) in (151) and plugging this into (141), we obtain that

$$\frac{d}{dt}K_q(h) \leq (c_1 - \frac{c_2^2}{4c_4})I_2 + \gamma^2 I_3 + (c_3 + 2\sqrt{-c_4}\gamma)J_{13} - \frac{c_2}{\sqrt{-c_4}}\gamma J_{23}. \tag{153}$$

Using the integration by parts relation (149) to eliminate I_2 term in (153), we obtain that

$$\begin{aligned} \frac{d}{dt}K_q(h) &\leq \left[c_3 + 2\sqrt{-c_4}\gamma - \frac{1}{(p+1)}(c_1 - \frac{c_2^2}{4c_4})\right] J_{13} \\ &+ \left[(\frac{q-n+2}{p+1})(c_1 - \frac{c_2^2}{4c_4}) - \frac{c_2}{\sqrt{-c_4}}\gamma\right] J_{23} + \gamma^2 I_3. \end{aligned} \tag{154}$$

Note that J_{13} can have either sign. Thus, we choose γ so that the multiple of it vanishes. This leads to the following choice of γ.

$$\gamma := \frac{\frac{1}{(p+1)}(c_1 - \frac{c_2^2}{4c_4}) - c_3}{2\sqrt{-c_4}}. \tag{155}$$

Plugging this choice of γ in (154) and also using the integration by parts identity (150), to eliminate J_{23} term, we finally have that

$$\frac{d}{dt}K_q(h) \leq C(p,q,n)I_3, \tag{156}$$

where $C(p,q,n)$ is a constant defined by

$$\begin{aligned} C(p,q,n) &:= \left((\frac{q-n+2}{p+1}(c_1 - \frac{c_2^2}{4c_4})) - \frac{c_2}{\sqrt{-c_4}}\left(\frac{q-n+2}{p+1}(c_1 - \frac{c_2^2}{4c_4})\right)\right)(\frac{q-n+3}{p+3}) \\ &+ (\frac{\frac{1}{(p+1)}(c_1 - \frac{c_2^2}{4c_4}) - c_3}{2\sqrt{-c_4}})^2. \end{aligned} \tag{157}$$

A simple calculation yields that if $p = 2$ and $n = 1$ then

$$C_1 := C(2,q,1) = -\frac{q^2}{360}(3 + 18q - 53q^2),$$

which exactly obeys the calculations in [25]. On the other hand for $p = 2, n = 2$ we have

$$C_2 := C(2,q,2) = -\frac{q^2}{360}(18 - 6q - 53q^2),$$

which works perfectly fine. If $0 \leq q < \frac{9+4\sqrt{15}}{53}$ then $C_1 \leq 0$ and for $0 \leq q < \frac{3\sqrt{107}-3}{53}$ then $C_2 < 0$. Hence, for the thin film equation i.e. $p = 2$ case in (67), we can show an entropy dissipation-entropy estimate for the physical cases $n = 1$ and $n = 2$. On the other hand we note that numerical calculations suggest that this can be done for a wider range of non integer values too. For the physical case $n = 3$ we do not expect to have such an estimate by the results of [61].

Unfortunately for $p = 3$ case it seems that we can include only the physical case $n = 2$ and we slightly miss $n = 1$ case. We note that non integer values can be included in this case too but we do not analyze these at the moment. We have given the calculations in [95] for $p = 3, n = 2$ case. In this case

$$C_3 := C(3, q, 2) = \frac{2371}{6912}q^4 + \frac{77}{432}q^3 - \frac{11}{144}q^2 - \frac{1}{54}q - \frac{1}{108}.$$

We have that for a critical value $q^* \in (0.4, 0.5)$ we have $C_3 \leq 0$ for $q \in [0, q^*]$.

Step 2 : *Now, we show that*

$$I_3 \geq N_q K_q(h), \tag{158}$$

where N_q is a positive constant.

Proof of Step 2 : Notice that

$$I_3 \geq \int \frac{|h_x|^7}{h^{q-n+4}} dx = \int (\frac{h_x^2}{h^q})^{7/2} \frac{1}{h^r} dx.$$

Letting $z = \frac{h_x^2}{h^q}$, and letting $v = h$, we have that

$$I_3 \geq \int z^{7/2} v^{-r} dx. \tag{159}$$

The function $(v, s) \to v^{7/2} s^{-r}$ is jointly convex if $r \leq 5/2$, so that by Jensen's inequality,

$$\begin{aligned} \frac{1}{2a}\int_{-a}^{a} z^{7/2} v^{-r} dx &\geq \left(\frac{1}{2a}\int_{-a}^{a} h dx\right)^{7/2} \left(\frac{1}{2a}\int_{-a}^{a} v dx\right)^{-r} \\ &= \frac{1}{2a(\int_{-a}^{a} h_0(x) dx)^r} (K_q(h))^{7/2}. \end{aligned} \tag{160}$$

Combination of (159) and (160) gives the result. Note that we have the following restriction on n

$$r = -5/2q - n + 4 \leq 5/2 \iff n \geq \frac{3}{2} - \frac{5}{2}q, \tag{161}$$

which is satisfied for small values of q satisfying the conclusion of Step 1.

Step 3 : *Consequence:*

For the physical cases $(p, n) \in \{(2,1),(2,2),(3,2)\}$ one can deduce using the above calculations that

$$K_q(h(t,x)) \leq \left[\frac{2}{5(Ct + \frac{2}{5}[K_q(0)]^{-5/2})}\right]^{2/5}. \tag{162}$$

This clearly gives an initial polynomial decay(like $t^{-2/5}$) of positive smooth solutions to the equilibrium and once $K_q(h)$ is small enough we can then use linearization to obtain an exponential decay. We provide the details of the linearization for thin film equation case (41) when $n = 1$.

We now apply the entropy dissipation results to quantify the rate of convergence provided by the dissipation of K_q. Given positive, continuous initial data h_0, let M denote the mean height; i.e.,

$$M = (2a)^{-1} \int_{-a}^{a} h_0(x)\mathrm{d}x\,, \tag{163}$$

so that M is the constant value of the equilibrium solution corresponding to h_0. For the classical solution h of the thin film equation (41) with $n = 1$ and with positive initial data h_0, clearly $(2a)^{-1} \int_{-a}^{a} h(t,x)\mathrm{d}x = M$ for all t.

It is not hard to see that when $K_q(h)$ is small, then so is $\|h - M\|_\infty$. This fact will be used below, and so we give a formal statement in the following lemma, which provides a sort of Poincare–Sobolev inequality for the functional K_q.

Lemma 13: For $p = 2$ and for any q with $0 < q < 2$, and any positive function h for which $K_q(h)$ is finite,

$$\|h - M\|_\infty^2 \leq 2a\left(M^{1-p/2} + (1 - p/2)(2a)^{1/2}(K_q(h))^{1/2}\right)^{2p/(2-p)} K_q(h). \tag{164}$$

After the polynomial decay, provided by the entropy dissipation, has gone on long enough, we reach a sufficiently small neighborhood of the equilibrium that it is possible to control the errors in linearization, and from this point onward, the decay is exponentially fast.

The following theorem, which makes this precise, is relatively easy to prove. However it is meaningful only on account of dissipation results and Lemma 13 that guarantee its applicability to solutions of our equation with initial data in a fairly general class.

Proof of Lemma 13:

Notice that with g defined by $g = (1-p/2)^{-1}h^{1-p/2}, K_q(h) = \int_{-a}^{a}(g_x)^2 dx$. Then since $g(b) = (1-p/2)^{-1}M^{1-p/2}$ for some b with $-a < b < a$,

$$\|g - (1-p/2)^{-1}M^{1-p/2}\|_\infty^2 \le \left(\int_{-a}^{a}|g_x|^2 dx\right)^2 \le 2aK_q(h). \tag{165}$$

Introduce $u = h - M$. Then

$$\begin{aligned} |g - (1-p/2)^{-1}M^{1-p/2}| &= (1-p/2)^{-1}|(M+h)^{1-p/2} - M^{1-p/2}| \\ &\ge |u|/W, \end{aligned} \tag{166}$$

where W is the maximum of $M^{p/2}$ and $\|M+u\|_\infty^{p/2}$. Since $(M+u)^{p/2} = h^{p/2} = ((1-p/2)g)^{p/(2-p)}$, we have from (165) that $W \le \left(M^{1-2/p} + (1-p/2)(2a)^{1/2}(K_q(h))^{1/2}\right)^{2p/(2-p)}$. Combining this with (166), which says that

$$\|h - M\|_\infty \le W\|g - (1-p/2)^{-1}M^{1-p/2}\|_\infty,$$

and then with (165), we obtain the result. □

Theorem 14: For any positive classical solution of the thin film equation (41) with $n = 1$, let M be the corresponding equilibrium value, and suppose the initial data h_0 is such that $K_q(h_0) < \infty$ for some q with $0 < q < (9 + 4\sqrt{15})/53$. Then for any $\epsilon > 0$, there is a finite time T_ϵ, explicitly computable in terms of M and $K_q(h_0)$ so that for all $t > T_\epsilon$, $\|h - M\|_\infty \le \epsilon$. Moreover, for all $t > T_\epsilon$, we have

$$K_0(h) \le K_0(h_0)e^{-(t-T_\epsilon)2(M-\epsilon)(\pi/a)^4}$$

in case we are using periodic boundary conditions, and

$$K_0(h) \le K_0(h_0)e^{-(t-T_\epsilon)2(M-\epsilon)(\pi/2a)^4}$$

in case we are using "no flux" boundary conditions.

Recall that $K_0(h) = \int h_x^2 \mathrm{d}x$, for this case, so Theorem 14 proves that this Sobolev norm decays to zero exponentially fast. Of course $\|h - M\|_\infty^2 \le (2a)K_0(h)$, and so Theorem 14 also ensures an exponential rate of convergence to the equilibrium in the uniform norm. As will be clear from the proof, which is based on linearization, the rates are essentially best possible, as one cannot hope for faster convergence than one would get from the linearized equation. While explicit exponential convergence to equilibrium was obtained in the L^1 norm earlier, the rates were considerably slower. To our knowledge, Theorem 14 provides the first proof that $\int h_x^2 \mathrm{d}x$ decreases to zero at any rate for any class of initial data that is not already close to equilibrium.

Proof of Theorem 14: Since if $\|h - M\|_\infty \le \epsilon$ then $h \ge M - \epsilon$, we have that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} K_0(h) &= -\quad 2\int_{-a}^{a} h(h_{xxx})^2 \mathrm{d}x \\ &\le -2(M-\epsilon)\int_{-a}^{a} (h_{xxx})^2 \mathrm{d}x\,. \end{aligned} \tag{167}$$

Under periodic boundary condition, h_x is orthogonal to the constant functions; i.e., the null space of the operator $-\mathrm{d}^2/\mathrm{d}x^2$ with periodic boundary conditions on $[-a, a]$. The least of the positive eigenvalues for this operator is $(\pi/a)^2$, so that under periodic boundary conditions, we obtain from (167) that

$$\frac{\mathrm{d}}{\mathrm{d}t} K_0(h) \le -2(M-\epsilon)\left(\frac{\pi}{a}\right)^4 K_0(h)\,.$$

Under the "no flux" boundary conditions, $h_x(\pm a) = h_{xxx}(\pm a) = 0$, h_x belongs to the domain of $-\mathrm{d}^2/\mathrm{d}x^2$ with Dirichlet boundary conditions on $[-a, a]$. Its smallest eigenvalue (in absolute value) is $(\pi/(2a))^2$. In this case we obtain from (167) that

$$\frac{\mathrm{d}}{\mathrm{d}t} K_0(h) \le -2(M-\epsilon)\left(\frac{\pi}{2a}\right)^4 K_0(h)\,.$$

$\square$

We note that one can use the linearization for other physical cases for which we have shown the dissipation of the entropy functional $K_q(h)$ above but we do not present the details for these cases.

3.6 Integral Estimates

Proposition 15 (Integral estimate for h_ϵ) : Let h_0 satisfy (85), P_ϵ be defined by (72) and let h_ϵ be the solution of the regularized problem (73) with the initial condition

$$h(0, x) = h_{0\epsilon}(x), \qquad x \in \Omega,$$

and with boundary conditions (69).

Suppose that

$$h_{0\epsilon} \in C^{\infty}([-a,a]), h_{0\epsilon} > 0, \quad \text{for} \;\; x \in [-a,a], h_{0\epsilon} \to h_0 \;\; in \;\; H^p((-a,a)) \;\; \text{as} \;\; \epsilon \to 0 \tag{168}$$

and moreover suppose that $h_{0\epsilon}$ satisfies the corresponding boundary conditions (69).

Let $\alpha \neq 0$ be a real number such that

$$\frac{p-1}{p} \leq \alpha + n \leq 2, \tag{169}$$

let $T > 0$, and let $\zeta \in C^4(\Omega)$ be a nonnegative function with support in $(-a,a)$. Assume either

$$h_0 > 0 \quad in \;\; supp(\zeta) \tag{170}$$

or $h_{0\epsilon}$ satisfies

$$h_{0\epsilon}(x) \geq h_0(x) + \epsilon^{\theta}, \;\; 0 < \theta \leq \frac{2}{5} \tag{171}$$

and h_0 satisfies

$$\int_{\Omega} \zeta^4 h_0^{\alpha+1}(x)dx < \infty, \;\; \alpha \neq -1 \tag{172}$$

$$\int_{\Omega} \zeta^4 |\ln(h_0(x))| dx < \infty, \;\; \alpha = -1. \tag{173}$$

Then, there exists constants C_1^* and C_2^* which are independent of ϵ such that

$$\int_{\Omega} \zeta^4 h_{\epsilon}^{\alpha+1}(t,x)dx \leq C_1^*, \quad 0 < t \leq T, \;\; \alpha \neq -1, \tag{174}$$

$$\int_{\Omega} \zeta^4 |\ln(h_{\epsilon}(t,x)| dx \leq C_2^*, \quad 0 < t \leq T, \;\; \alpha = -1. \tag{175}$$

If γ is a real number satisfying

$$\gamma_1 \leq \gamma \leq \gamma_2, \tag{176}$$

where

$$\gamma_1 := \frac{(\alpha + n + p - 1) - \sqrt{(\alpha + n - 2)(p - 1 - p(\alpha + n))}}{(p+1)}, \tag{177}$$

and

$$\gamma_2 := \frac{(\alpha + n + p - 1) + \sqrt{(\alpha + n - 2)(p - 1 - p(\alpha + n))}}{(p+1)}, \tag{178}$$

then

$$\int_0^t \int_\Omega \zeta^4 h_\epsilon^{\alpha+n-2\gamma+1}(h_\epsilon^\gamma)_{xx}^2 (h_{\epsilon x}^2)^{p/2-1} dxdt \le C_3^*, \tag{179}$$

and if

$$\frac{p-1}{p} < \alpha + n < 2,$$

then

$$\int_0^t \int_\Omega \zeta^4 h_\epsilon^{\alpha+n-3} h_{\epsilon x}^4 (h_{\epsilon x}^2)^{p/2-1} dxdt \le C_4^*, \tag{180}$$

where C_3^* and C_4^* are positive constants independent of ϵ.

Remark 1.

If the conditions of the proposition are satisfied and h is a solution of the corresponding limiting case where $\epsilon \to 0$, then, by *Fatou's lemma* one deduces that

$$\int_\Omega \zeta^4 h^{\alpha+1}(t,x) dx < \infty, \quad t > 0, \quad \alpha \neq -1, \tag{181}$$

$$\int_\Omega \zeta^4 |\ln(h(t,x))| dx < \infty, \quad t > 0, \quad \alpha = -1. \tag{182}$$

Remark 2.

The inequality (179) becomes

$$\int_0^t \int_\Omega \zeta^4 h_\epsilon^{1/p} (h_\epsilon^{(p-1)/p})_{xx}^2 dxdt \le C_3^* \tag{183}$$

if $\alpha + n = \frac{p-1}{p}, \gamma = \frac{p-1}{p}$ and $n \neq \frac{p-1}{p}(i.e., \alpha \neq 0)$, and it becomes

$$\int_0^t \int_\Omega \zeta^4 h_\epsilon h_{\epsilon xx}^2 dxdt \le C_3^* \tag{184}$$

if $\alpha + n = 2, \gamma = 1$ and $n \neq 2(i.e., \alpha \neq 0)$.

Proof :

Define the following function

$$g_\epsilon(s) := -\int \frac{\alpha r^{\alpha+n-1}}{P_\epsilon(r)} dr = c_1 s^{c_2} + s^\alpha - c_1 A^{c_2} - A^\alpha, \tag{185}$$

where the constants c_1 and c_2 are given by

$$c_1 := \frac{\alpha\epsilon}{n+\alpha-2-\frac{p}{(p-1)}}$$

$$c_2 := n+\alpha-2-\frac{p}{(p-1)}.$$

Now define,

$$G_\epsilon(s) := -\int_s^A g_\epsilon(r)dr, \tag{186}$$

where $A > \max h_\epsilon$, and $0 < s < A$. As $h_\epsilon > 0$ the functions g_ϵ and G_ϵ are well defined.

Let $\alpha \neq -1$. Multiplying the equation (73) by $\zeta^4 g_\epsilon(h)$ and integrating by parts, for any $t \in (0, T]$, one has that(note that we represent the solution of (73) by h to simplify notation. At the end of the proof we will return to the original notation.)

$$\begin{aligned}
\tfrac{1}{\alpha} & \int_\Omega \zeta^4 G_\epsilon(h(t,x))dx - \frac{1}{\alpha}\int_\Omega \zeta^4 G_\epsilon(h_{0\epsilon}(x))dx \\
= \ & (p-1)\int_0^t\int_\Omega \zeta^4 h^{\alpha+n-1}h_x[(h_x^2)^{p/2-1}h_{xx}]_x dx + \frac{(p-1)}{\alpha}\int_0^t\int_\Omega (\zeta^4)_x g_\epsilon(h)P_\epsilon(h)[(h_x^2)^{p/2-1}h_{xx}]_x dx \\
=: \ & L_1 + L_2.
\end{aligned} \tag{187}$$

We can integrate by parts and write L_1 as

$$\begin{aligned}
L_1 = \ & - \ (p-1)\int_0^t\int_\Omega (\zeta^4)_x h^{\alpha+n-1}(h_x^2)^{p/2-1}h_x h_{xx}dxdt \\
& - \ (p-1)(\alpha+n-1)\int_0^t\int_\Omega \zeta^4 h^{\alpha+n-2}(h_x^2)^{p/2-1}h_x^2 h_{xx}dxdt \\
& - \ (p-1)\int_0^t\int_\Omega \zeta^4 h^{\alpha+n-1}(h_x^2)^{p/2-1}h_{xx}^2 dxdt \\
& =: \ -c_1 L_{1,1} - c_2 L_{1,2} - c_1 L_{1,c}.
\end{aligned} \tag{188}$$

To benefit fully from the sign of the term $L_{1,c}$ in (188) we use the following substitution, which was used in [8],

$$h_{xx}^2 = \frac{1}{\gamma^2}h^{2-2\gamma}(h^\gamma)_{xx}^2 - (\gamma-1)^2 h^{-2}h_x^4 - 2(\gamma-1)h^{-1}h_x^2 h_{xx}, \tag{189}$$

where γ is a positive constant.

Using (189) in (188) and collecting the likely terms together we obtain that

$$\begin{aligned}
L_1 = &- c_1 L_{1,1} - c_1(\alpha + n - 2\gamma + 1) L_{1,2} \\
&- \frac{(p-1)}{\gamma^2} \int_0^t \int_\Omega \zeta^4 h^{\alpha+n-2\gamma+1} (h_x^2)^{p/2-1} (h^\gamma)_{xx}^2 dxdt \\
&+ (p-1)(\gamma-1)^2 \int_0^t \int_\Omega \zeta^4 h^{\alpha+n-3} (h_x^2)^{p/2-1} h_x^4 dxdt \\
=: & -c_1 L_{1,1} - c_3 L_{1,2} - c_4 L_{1,3} + c_5 L_{1,4}, \qquad (190)
\end{aligned}$$

where in (190) we define the quantities on the right hand side according to the occurrence of the quantities on the left hand side.

Before proceeding further we prove the following result.

Lemma 16 : One has the following integration by parts relations.

$$\begin{aligned}
L_{1,2} &= -\frac{(\alpha+n-2)}{(p+1)} L_{1,4} - \frac{1}{(p+1)} \int_0^t \int_\Omega (\zeta^4)_x h^{\alpha+n-2} (h_x^2)^{p/2-1} h_x^3 dxdt \\
&=: -c_6 L_{1,4} - c_7 L_{1,5}. \qquad (191)
\end{aligned}$$

$$\begin{aligned}
L_{1,1} &= \frac{(\alpha+n-1)}{p} L_{1,5} - \frac{1}{p} \int_0^t \int_\Omega (\zeta^4)_{xx} h^{\alpha+n-1} (h_x^2)^{p/2-1} h_x^2 dxdt \\
&=: -c_8 L_{1,5} - \frac{1}{p} L_{1,6}. \qquad (192)
\end{aligned}$$

Proof: This is a straight forward calculation.

□

Using (191) and (192) and also collecting the likely terms together we finally obtain that

$$L_1 = -c_4 L_{1,3} - c(\alpha+n,\gamma) L_{1,4} + R_1. \qquad (193)$$

Here,

$$c(\alpha+n,\gamma) := -[(p-1)(\gamma-1)^2 + \frac{(p-1)}{(p+1)}(\alpha+n-2\gamma+1)(\alpha+n-2)] \qquad (194)$$

and

$$R_1 = K_1 L_{1,5} + (\frac{p-1}{p}) L_{1,6}, \qquad (195)$$

where

$$K_1 := \frac{(p-1)}{p}(\alpha+n-1) - \frac{(p-1)}{(p+1)}(2\gamma - 1 - (\alpha+n)). \tag{196}$$

One can easily obtain that

$$c(\alpha+n,\gamma) = 0 \Leftrightarrow \gamma = \gamma_1 \;\; or \;\; \gamma = \gamma_2, \tag{197}$$

where γ_1 and γ_2 are given by (177) and (178) respectively. Moreover,

$$c(\alpha+n,\gamma) \geq 0 \Longleftrightarrow \gamma_1 \leq \gamma \leq \gamma_2. \tag{198}$$

Now we start estimating L_2. For this purpose we write

$$\frac{1}{\alpha}P_\epsilon(h)g_\epsilon(h) = m(h) + c_\epsilon P_\epsilon(h), \tag{199}$$

where

$$m(h) := \frac{1}{\alpha}P_\epsilon(h)[\frac{\alpha\epsilon}{(\alpha+n-2-p/(p-1))}h^{(\alpha+n-2-p/(p-1))} + h^\alpha], \tag{200}$$

and

$$c_\epsilon := -\frac{1}{\alpha}A^\alpha - \frac{\epsilon}{(\alpha+n-2-p/(p-1))}A^{(\alpha+n-2-p/(p-1))}. \tag{201}$$

Using these we can rewrite L_2 as

$$\begin{aligned} L_2 &= \int_0^t \int_\Omega (\zeta^4)_x m(h)[(h_x^2)^{p/2-1}h_{xx}]_x dxdt \\ &+ c_\epsilon \int_0^t \int_\Omega (\zeta^4)_x P_\epsilon(h)[(h_x^2)^{p/2-1}h_{xx}]_x dxdt \\ &=: L_{2,1} + c_\epsilon L_{2,2}. \end{aligned} \tag{202}$$

One keeps the second term and integrates by parts the first term to obtain

$$L_2 = c_\epsilon L_{2,2} - \int_0^t \int_\Omega [(\zeta^4)_x m(h)]_x [(h_x^2)^{p/2-1}h_{xx}]dxdt =: c_\epsilon L_{2,2} - L_{2,3}. \tag{203}$$

To proceed further we need to prove the following result.

Lemma 17 : One has the following integration by parts relation.

$$
\begin{aligned}
-L_{2,3} &= \frac{1}{(p-1)}\int_0^t\int_\Omega (\zeta^4)_{xxx} m(h)(h_x^2)^{p/2-1} h_x dxdt \\
&+ \frac{2p-1}{p(p-1)}\int_0^t\int_\Omega (\zeta^4)_{xx} m'(h)(h_x^2)^{p/2-1} h_x^2 dxdt \\
&+ \frac{1}{p}\int_0^t\int_\Omega (\zeta^4)_{x} m''(h)(h_x^2)^{p/2-1} h_x^3 dxdt \\
&=: c_4L_{2,4} + c_5L_{2,5} + c_6L_{2,6}. \qquad (204)
\end{aligned}
$$

Proof: This is a straight forward calculation.

□

Using (204) and also collecting likely terms together, we finally deduce that

$$
\begin{aligned}
L_2 &= c_\epsilon L_{2,2} + c_5L_{2,5} + c_6L_{2,6} \\
&- \frac{1}{(p-1)}\int_0^t\int_\Omega (\zeta^4)_{xxxx}(h_x^2)^{p/2-1} M_1(h) dxdt \\
&- \frac{(p-2)}{(p-1)}\int_0^t\int_\Omega (\zeta^4)_{xxx}(h_x^2)^{p/2-2} h_x h_{xx} M_1(h) dxdt \\
&=: c_\epsilon L_{2,2} + c_5L_{2,5} + c_6L_{2,6} - c_7L_{2,7} - c_8L_{2,8}, \qquad (205)
\end{aligned}
$$

where

$$
M_1(h) := \int_0^h m(r)dr.
$$

Let $s \in (0, A)$, by considering the definitions of $m(h)$ and P_ϵ, we can deduce the following estimates

$$
|m(s)| \le K_2 s^{n+\alpha}, \qquad |m'(s)| \le K_3 s^{n+\alpha-1}; \qquad (206)
$$

$$
|m''(s)| \le K_4 s^{n+\alpha-2}, \qquad |M_1(s)| \le K_5 s^{n+\alpha+1}. \qquad (207)
$$

Using these estimates we will bound $R_1 + L_2$. But first we state the following result.

Lemma 18 : One has the following integration by parts relation.

$$
\begin{aligned}
L_{2,8} &= -\frac{1}{(p-2)}L_{2,7} - \frac{1}{(p-2)}\int_0^t\int_\Omega (\zeta^4)_{xxx}(h_x^2)^{p/2-2}m(h)h_x^3\,dxdt \\
&=: -\frac{1}{(p-2)}\left(L_{2,7}+L_{2,9}\right). \qquad (208)
\end{aligned}
$$

Proof: This is a straight forward calculation.

□

Using these, together with the smoothness of ζ, we obtain that

$$
\begin{aligned}
|R_1 + L_2| &\leq |c_\epsilon|\int_0^t\int_\Omega (\zeta^4)_x P_\epsilon(h)[(h_x^2)^{p/2-1}h_{xx}]_x\,dxdt \\
&+ C_2\int_0^t\int_\Omega \zeta^2 h^{\alpha+n-1}(h_x^2)^{p/2-1}h_x^2\,dxdt \\
&+ C_3\int_0^t\int_\Omega \zeta^3 h^{\alpha+n-2}(h_x^2)^{p/2-1}|h_x^3|\,dxdt \\
&+ C_4\int_0^t\int_\Omega h^{\alpha+n+1}(h_x^2)^{p/2-1}\,dxdt \\
&+ C_5\int_0^t\int_\Omega \zeta h^{\alpha+n}(h_x^2)^{p/2-1}|h_x|\,dxdt \\
&=: |c_\epsilon|L_{2,r,1} + C_2L_{2,r,2} + C_3L_{2,r,3} + C_4L_{2,r,4} + C_5L_{2,r,5}. \qquad (209)
\end{aligned}
$$

The first term in (209) is uniformly bounded by the dissipation result and uniform boundedness of the terms c_ϵ and $P_\epsilon(h)$. Indeed, by the Hölder's inequality we have that

$$
\begin{aligned}
|c_\epsilon|L_{2,r,1} &\leq |c_\epsilon|\left(\int_0^t\int_\Omega P_\epsilon(h)([(h_x^2)^{p/2-1}h_{xx}]_x)^2\,dxdt\right)^{1/2}\left(\int_0^t\int_\Omega |(\zeta^4)_x|^2 P_\epsilon(h)\,dxdt\right)^{1/2} \\
&\leq C'|c_\epsilon|\left(\int_0^t\int_\Omega |(\zeta^4)_x|^2 P_\epsilon(h)\,dxdt\right)^{1/2} \leq C'', \qquad (210)
\end{aligned}
$$

where we have used the dissipation result, and uniform boundedness of c_ϵ, $P_\epsilon(h)$ and $|(\zeta^4)_x|$.

Now, we will show that last two terms in (209) are uniformly bounded. Indeed, by the Hölder's inequality we have that

$$
C_4L_{2,r,4} \leq C_4\left(\int_0^t\int_\Omega (h_x^2)^{p/2}\,dxdt\right)^{(p-2)/p}\left(\int_0^t\int_\Omega h^{\frac{p}{2}(\alpha+n+1)}\,dxdt\right)^{2/p} \leq C, \qquad (211)
$$

where we have used the energy dissipation and the fact that $\frac{p}{2}(\alpha+n+1) > 0$.

Similarly, we have that

$$C_5 L_{2,r,5} \le C_5 \left(\int_0^t \int_\Omega (h_x^2)^{p/2} dxdt \right)^{(p-1)/p} \left(\int_0^t \int_\Omega \zeta h^{p(\alpha+n)} dxdt \right)^{1/p} \le C, \tag{212}$$

where again we used the energy dissipation and the fact that $p(\alpha + n) > 0$. Collecting these results together and using the fact that ζ is a smooth bounded function and h_ϵ is a smooth, positive function(so that it is bounded from below), we deduce from (209) that

$$\begin{aligned} |R_1 + L_2| &\le C_1' \\ &+ c_6 \int_0^t \int_\Omega (h_x^2)^{p/2-1} h_x^2 dxdt \\ &+ c_7 \int_0^t \int_\Omega (h_x^2)^{p/2-1} |h_x^3| dxdt \\ &=: C_1' + c_6 L_{2,r,6} + c_7 L_{2,r,7}, \end{aligned} \tag{213}$$

where C_1', c_6, c_7 are constants. To bound the last two terms in (213), we let $\beta = 0$ in (132) and we deduce that

$$\int_0^t \int_\Omega [(h_x^2)_x^{p/4}]^2 dxdt < \infty. \tag{214}$$

Using this we obtain that

$$\int_0^t \|h_x\|_{L^\infty}^p dt = \int_0^t \|(h_x^2)^{p/2}\|_{L^\infty}^2 dt \le c \int_0^t \int_\Omega [(h_x^2)_x^{p/4}]^2 dxdt \le C. \tag{215}$$

Now, rewriting the the second term in (213) we have that

$$|c_6| L_{2,r,6} = |c_6| \int_0^t \int_\Omega |h_x|^p dxdt \le |c_6| \int_0^t \int_\Omega \|h_x\|_{L^\infty}^p dxdt \le 2aC. \tag{216}$$

Similarly, the last term in (213) can be bounded by

$$|c_7| L_{2,r,7} = \int_0^t \int_\Omega |h_x|^{p+1} dxdt \le \int_0^t \|h_x\|_{L^\infty} (\int_\Omega |h_x|^p dx) dt \le C \int_0^t \|h_x\|_{L^\infty} dt \le C_1. \tag{217}$$

Collecting what we have obtained so far we finally deduce that

$$\frac{1}{\alpha} \int_\Omega \zeta^4 G_\epsilon(h(x,t)) dx + c_4 L_{1,3} + c(\alpha+n,\gamma) L_{1,4} \le \frac{1}{\alpha} \int_{-a}^a \zeta^4 G_\epsilon(h_{0\epsilon}(x)) dx + \bar{K}, \tag{218}$$

where $\bar{K}$ is a constant. Notice that by assumption

$$\frac{1}{\alpha}\int_\Omega \zeta^4 G_\epsilon(h_{0\epsilon}(x))dx \leq K', \tag{219}$$

where K' is a constant independent of ϵ. Finally using (219) in (218) and the definition of G_ϵ we deduce that(now we start using the original notation...)

$$\begin{aligned} c_4 L_{1,3} + c(\alpha+n,\gamma)L_{1,4} + \frac{\epsilon}{c_*(c_*-1)}\int_\Omega \zeta^4 h_\epsilon^{c_*}(t,x)dx \quad &\leq \quad -\frac{1}{\alpha(\alpha+1)}\int_\Omega \zeta^4 h_\epsilon^{\alpha+1}(t,x)dx + \tilde{K} \\ &\Longleftrightarrow \\ c_4 L_{1,3} + c(\alpha+n,\gamma)L_{1,4} + \tilde{c_*}\tilde{L} \quad &\leq \quad -c_\alpha L_\alpha + \tilde{K}, \end{aligned} \tag{220}$$

where $c_* = \alpha+n-\frac{p}{(p-1)}-1$ and $\tilde{K}$ is a constant independent of ϵ. Note that L_α is uniformly bounded if $\alpha+1>0$ and has a negative coefficient when $\alpha+1<0$, we then deduce that

$$c_4 L_{1,3} + c(\alpha+n,\gamma)L_{1,4} + \tilde{c_*}\tilde{L} + |c_\alpha| L_\alpha \leq K^* \tag{221}$$

where K^* is a constant independent of ϵ. Since the terms on the left hand side of (221) are nonnegative, we obtain that for $t \in (0,T]$

$$L_\alpha \leq C_1^*, \tag{222}$$

where C_1^* is a constant independent of ϵ.

Moreover, we also obtain that

$$L_{1,3} \leq C_3^*, \tag{223}$$

where again C_3^* is a constant independent of ϵ. Choosing α, n and γ so that $c(\alpha+n,\gamma)>0$, we finally deduce that

$$L_{1,4} \leq C_4^*, \tag{224}$$

where C_4^* is a constant independent of ϵ.

One can modify the calculations for $\alpha \neq 1$ and obtain $\int_\Omega \zeta^4 |\ln(h_\epsilon(t,x))|dx \leq C_2^*$ for $t \in (0,T]$.

□

Corollary 19 (A useful integral estimate) : Let $\alpha \neq 0$ and γ be real numbers satisfying

$$\frac{p-1}{p} < \alpha + n < 2,$$

$$\alpha + n + p - 1 < 3\gamma < \alpha + n + p - 1 + \sqrt{(\alpha + n - 2)(p - 1 - p(\alpha + n))}. \qquad (225)$$

Let $T > 0$, and assume that $h_0, h_{0\epsilon}, P_\epsilon, \zeta$ satisfy the conditions of the proposition and let h_ϵ be the solution of the regularized problem (73) with the initial condition

$$h(0, x) = h_{0\epsilon}(x), \qquad x \in \Omega,$$

and with boundary conditions (69). Then, there exists a constant C, independent of ϵ, such that

$$\int_0^t \int_\Omega \zeta^4 |(|(h_\epsilon^\gamma)_x|^{(4-q)/q})_x|^q (h_{\epsilon x}^2)^{p/2-1} dxdt \leq C, \qquad (226)$$

where

$$q = \frac{4\gamma - 1 - n - \alpha}{\gamma} \in (1, 2). \qquad (227)$$

Proof. By the proposition 16 h_ϵ satisfies the integral estimates. We can rewrite (179) as

$$\int_0^t \int_\Omega \zeta^4 (h_\epsilon^\gamma)_x^4 h_\epsilon^{\alpha+n+1-4\gamma} (h_{\epsilon x}^2)^{p/2-1} dxdt \leq C_3^*.$$

Note also that we write (226) in the given form as constants were worked out in [8]. This simplifies some of the calculations below.

We will choose $q \in (1, 2)$ and $\lambda > 0$ below and we set $p^* = \frac{4-q}{q}$. We apply the Hölder's inequality with exponents $p' = 2/q$ and $q' = 2/(2-q)$ to obtain that

$$\int_0^t \int_\Omega \zeta^4 |(|(h_\epsilon^\gamma)_x|^{p^*})_x|^q (h_{\epsilon x}^2)^{p/2-1} dxdt$$

$$\leq C \left(\int_0^t \int_\Omega \zeta^4 |(h_\epsilon^\gamma)_{xx}|^2 h_\epsilon^{2\gamma/q} (h_{\epsilon x}^2)^{p/2-1} dxdt \right)^{q/2} \left(\int_0^t \int_\Omega \zeta^4 |(h_\epsilon^\gamma)_x|^4 h_\epsilon^{-2\gamma/(2-q)} (h_{\epsilon x}^2)^{p/2-1} dxdt \right)^{(2-q)/q} \qquad (228)$$

Hence, to proceed further we need to show that we can choose $\lambda > 0$ and $1 < q < 2$ such that

$$-\lambda \frac{2}{2-q} \geq \alpha + n + 1 - 4\gamma, \;\; \frac{2\lambda}{q} \geq \alpha + n + 1 - 2\gamma. \qquad (229)$$

Setting q as in (227) and

$$\lambda = \frac{2-q}{2}(4\gamma - 1 - n - \alpha),$$

we see that in order to show that (229) is satisfied we need to assume (225). This completes the proof.

□

Corollary 20 (The case $T = +\infty$) : Let $\alpha \neq 0$ and $n > 0$ satisfy

$$\frac{p-1}{p} \leq \alpha + n \leq 2,$$

and let γ satisfy

$$\gamma_1 \leq \gamma \leq \gamma_2, \tag{230}$$

where

Let $h_0, h_{0\epsilon}$ and P_ϵ satisfy the conditions given in the Proposition 16, and let $\zeta = 1$ in $[-a, a]$. If h_ϵ is the solution of (73) with no flux boundary conditions then there exist constants C_1, C_2 which are independent of ϵ such that

$$\int_0^\infty \int_\Omega h_\epsilon^{\alpha+n-2\gamma+1} (h_\epsilon^\gamma)_{xx}^2 (h_{\epsilon x}^2)^{p/2-1} dxdt \leq C_1, \tag{231}$$

and if $\frac{p-1}{p} < \alpha + n < 2$, then

$$\int_0^\infty \int_\Omega h_\epsilon^{\alpha+n-3} h_{\epsilon x}^4 (h_{\epsilon x}^2)^{p/2-1} dxdt \leq C_2. \tag{232}$$

If α, n and γ satisfy (227), then there exists a constant C_3, which is independent of ϵ such that

$$\int_0^\infty \int_\Omega |(|(h_\epsilon^\gamma)_x|^{(4-q)/q})_x|^q (h_{\epsilon x}^2)^{p/2-1} dxdt \leq C_3, \tag{233}$$

where q is defined by (227).

Proof. The proof of the Proposition 16 and the proof of the Corollary 20 work if we take $Q = (0, \infty) \times \Omega$ instead of $Q_T = (0, T) \times \Omega$ and take $\zeta \equiv 1$ in $[-a, a]$.

□

Lemma 21 (Integral estimates for h) : Let $\alpha \neq 0$ and $n > 0$ satisfy

$$\frac{p-1}{p} < \alpha + n < 2.$$

Let h_0 satisfy the conditions of the Proposition 16 above with $\zeta \equiv 1$ in $[-a, a]$, and let h be a solution of the problem (67), (68) and (69). Then

$$\int_0^\infty \int_\Omega h^{\alpha+n-3} h_x^4 (h_x^2)^{p/2-1} dxdt < \infty, \tag{234}$$

and for almost every $t > 0$ there exists a constant $C(t) < \infty$ such that

$$\text{if } h(t, y) = 0 \text{ for some } y \in [-a, a], \text{ then}$$

$$|h(t, x)| \leq C(t)|x - y|^m, for \;\; x \in [-a, a], \tag{235}$$

where

$$m := \frac{p+1}{\alpha + n + p - 1}.$$

Moreover, if α, n and γ satisfy (225), then

$$\int_0^\infty \int_\Omega |(|(h^\gamma)_x|^{(4-q)/q})_x|^q (h_x^2)^{p/2-1} dxdt < \infty, \tag{236}$$

where q is defined by (227).

Proof. Notice that we write (236) in the above form because the constants are already worked out in [8]. By (87) and (180), we deduce that

$$(h_{\epsilon_k}^{\tau_1})_x \to (h^{\tau_1})_x, \quad \text{weakly} \quad \text{in} \quad L_{loc}^{p+2}(\bar{Q}) \quad \text{as} \quad \epsilon_k \to 0,$$

where

$$\tau_1 := \frac{\alpha + n + p - 1}{p + 2}. \tag{237}$$

This shows that (234) holds.

Claim1. We have the following convergence result.

$$(h_{\epsilon_k}^\tau)_x \to (h^\tau)_x$$

strongly in $L_{loc}^{p+2}(\bar{Q})$, as $\epsilon_k \to 0$ for any $\tau > \tau_1$, where τ_1 is given by (237)

Proof of Claim1. We have

$$|(h_{\epsilon_k}^\tau)_x - (h^\tau)_x| \leq |h_{\epsilon_k}^{\tau-\tau_1} - h^{\tau-\tau_1}||(h_{\epsilon_k}^{\tau_1})_x| + h^{\tau-\tau_1}|(h_{\epsilon_k}^{\tau_1})_x - (h^{\tau_1})_x|. \tag{238}$$

Notice that (87) and (180) imply that the first term on the right hand side of (242) converges strongly to 0 in $L^{p+2}_{loc}(\bar{Q})$.

For the second term we fix T and consider the sets

$$Q^1_r := \{(t,x) \in Q_T : 0 \leq h \leq r\}$$

and

$$Q^2_r := \{(t,x) \in Q_T : h > r\},$$

where r is an arbitrary positive number. Since $\tau > \tau_1$ we deduce from (234) and (180) that

$$\iint_{Q^1_r} h^{(p+2)(\tau-\tau_1)} |(h^{\tau_1}_{\epsilon_k})_x - (h^{\tau_1})_x| dxdt \to 0$$

uniformly in ϵ_k as $r \to 0$. On the other hand, since the derivatives of h_{ϵ_k} converge uniformly on compact subsets of the set Q^2_r we deduce immediately that

$$\iint_{Q^2_r} h^{(p+2)(\tau-\tau_1)} |(h^{\tau_1}_{\epsilon_k})_x - (h^{\tau_1})_x| dxdt \to 0 \;\; as \;\; \epsilon_k \to 0.$$

Hence the proof of the claim is complete.

Now note that we can rewrite (234) as follows:

$$\iint_Q |(((h^r)_x)^b)_x|^q dxdt \leq C < \infty, \tag{239}$$

where

$$r := 1 + \frac{(1-\frac{1}{\gamma})(\alpha + n + 1)}{p + 2 - q}, \tag{240}$$

$$b := \frac{p+2-q}{q}, \tag{241}$$

and q is given by (227).

As $\gamma > \tau_1$ and $\frac{2+p-q}{q} < p+2$, by claim1 we deduce that $(h^{\gamma}_{\epsilon_k})_x \to (h^{\gamma})_x$ strongly in $L^{\frac{p+2-q}{q}}_{loc}(\bar{Q})$ as $\epsilon_k \to 0$. Hence, we may pass to the limit as $\epsilon_k \to 0$ and deduce that (234) holds.

Since $\frac{p-1}{p} < \alpha + n < 2$ there exists γ as assumed in the statement of lemma. Also , by (239) and the Sobolev Embedding theorem, for almost every $t > 0$, there exists a constant $C_1(t) < \infty$ such that

$$||(h^r)_x|^b(t,x) - |(h^r)_x|^b(t,y)| \leq C_1(t)|x-y|^{\frac{q-1}{q}}, \forall x, y \in [-a, a],$$

where b and r are given by (241) and (240) respectively. Hence, assuming $h(t, y) = 0$ and integrating this inequality yield (235).

□

3.7 Regularity and Large-Time Behavior of Solutions

Proof of Theorem 2

We note that since $0 < n < 3$ we may choose $\alpha > -1 (\alpha \neq 0)$ satisfying the conditions necessary for the integral estimates proved above. Moreover, it is enough to show that

if $0 < b < b_n$, then there is a constant $\tau > 0$ such that for almost every $t > 0$ there exists $C(t) < \infty$ such that

if $h(t, y) = 0$, then

$$|(h^{1/b})_x(t, x)| \leq C_(t)|x - y|^\tau, \quad x \in \Omega. \tag{242}$$

By the integral estimates for the solution h of the problem (67), (68) and (69) we deduce that if $h(t, y) = 0$, then for some finite constant C, we have that

$$|h(t, x)| \leq C(t)|x - y|^{\frac{p+1}{\alpha+n+p-1}}, \tag{243}$$

where $\frac{p-1}{p} < \alpha + n < 2$. We also have from the proof of Lemma 21 that if $h(t, y) = 0$, then

$$|(h^r)_x| \leq C(t)|x - y|^{\frac{q-1}{bq}}, \tag{244}$$

where

$$b = \frac{p + 2 - q}{q}, \qquad r = 1 + \frac{(1 - \frac{1}{\gamma})(\alpha + n + 1)}{p + 2 - q},$$

γ is a positive constant which was extensively used in the calculations above and q is given by

$$q = \frac{4\gamma - (\alpha + n + 1)}{\gamma} \in (1, 2). \tag{245}$$

Combining (243) and (244) and assuming $br < 1$, we deduce that

$$\begin{aligned} |(h^{1/b})| &\leq C(t)|x - y|^{(\frac{1}{br}-1)\frac{r(p+1)}{\alpha+n+p-1}}|x - y|^{\frac{q-1}{p+2-q}} \\ &\leq C(t)|x - y|^{\frac{1}{b}(\frac{p+1}{\alpha+n+p-1})-1}. \end{aligned} \tag{246}$$

Therefore to proceed we have to prove that

Given $0 < n < 3$ *and* $0 < b < b_n$, *we can choose* $\alpha > -1 (\alpha \neq 0)$ *and* γ *satisfying*

(i) $br < 1$,

(ii) $\frac{p-1}{p} < \alpha + n < 2$,

(iii) $\alpha + n + p - 1 < 3\gamma < \alpha + n + p - 1 + \sqrt{(\alpha + n - 2)(p - 1 - p(\alpha + n))}$.

Notice that once we prove these, then (242) follows with $\tau = \frac{1}{b}\frac{p+1}{\alpha+n+p-1} - 1$. We begin by choosing γ by

$$\gamma = \frac{1}{b_n} + \nu = \begin{cases} \frac{p-1}{p} + \nu & \text{if } 0 < n \leq 3\frac{(p-1)}{p} \\ \frac{n}{3} + \nu & \text{if } 3\frac{(p-1)}{p} \leq n < 3, \end{cases}$$

Note that (i) is satisfied if

$$0 < \nu < d_n := \min(\nu_1, \nu_2),$$

where

$$\nu_1 := \frac{\alpha + n + 1}{(\alpha + n + 1)b - p + 2} - \frac{p-1}{p}, \qquad \nu_2 := \frac{\alpha + n + 1}{(\alpha + n + 1)b - p + 2} - \frac{n}{3}.$$

We now fix $\nu \in (0, d_n)$ and we choose α by

$$\alpha = \frac{3}{b_n} - n - 1 + \mu = \begin{cases} 3\frac{p-1}{p} - n - 1 + \mu & \text{if } 0 < n \leq 3\frac{(p-1)}{p} \\ -1 + \mu & \text{if } 3\frac{(p-1)}{p} \leq n < 3, \end{cases},$$

where $\mu > 0$ so that $\alpha \neq 0$. Clearly, $\alpha > -1$. $\alpha + n > \frac{p-1}{p}$ is satisfied as $p \geq 2$. On the other hand, $\alpha + n + p - 1 < 3\gamma$ is satisfied if $\mu < 3\nu + 2 - p$. As $p \geq 2$ this says $\mu < 3\nu = 3(1 - \frac{1}{b_n})$. Thus, $\alpha + n < 2$. Finally, we note that the last inequality in (iii) is valid if $0 < \mu < 3\nu + 2 - p$ such that $3\nu + 2 - p - \mu$ is small enough.

This completes the proof. □

Proof of Theorem 3: From the integral estimates proved above for the equation under consideration we deduce that

$$\int_0^T \int_{-a}^{a} |(h^M)_x|^R dx < \infty, \tag{247}$$

where

$$M := \frac{\alpha + n - 3}{p + 2} + 1, \qquad R := p + 2.$$

From this, if we define

$$K(t) := \max_{[-a,a]} h^{\frac{\alpha+n+p-1}{p+2}}(t, \cdot) - \min_{[-a,a]} h^{\frac{\alpha+n+p-1}{p+2}}(t, \cdot),$$

we deduce that $K \in L^1(\mathbb{R})$. Thus, $K(t) \to 0$ as $t \to \infty$ as $K(t)$ is uniformly continuous in $\mathbb{R}^+$(by uniform Hölder continuity of u). Hence, we deuce that

$$\max_{[-a,a]} h(t, \cdot) - \min_{[-a,a]} h(t, \cdot) \to 0 \ \ as \ t \to \infty. \tag{248}$$

Combining (248) with the mass conservation we conclude the theorem.

□

3.8 Support properties of Solutions

Proof of Theorem 4:

(i): Let $h_0(x) > 0$ in some interval $(b, c) \subset (-a, a)$ and let ζ be a smooth nonnegative function with support in (b, c). First assume that $n > 1 + \frac{(p-1)}{p}$. Then, we can find an $\alpha < -1$ satisfying $\frac{p-1}{p} \leq \alpha + n < 2$ such that

$$\int_\Omega \zeta^4 h^{\alpha+1}(t, x)dx < \infty, \ \ for \ \ t > 0. \tag{249}$$

Since $\alpha < -1$ we obtain the result in this case.

Now, if $n = 1 + \frac{(p-1)}{p}$, then we choose $\alpha = -1$ and we know that

$$\int_\Omega \zeta^4 |\ln(h(t, x))| dx \ \ for \ \ t > 0 \tag{250}$$

But this implies the result.

(ii): Since $h_0(x_0) > 0$ and h_0 is continuous there exists a $\delta > 0$ such that $h_0(x) > 0$ for $x \in (x_0 - 2\delta, x_0 + 2\delta)$. Let $\zeta(x)$ be a smooth nonnegtaive function with support in $(x_0 - 2\delta, x_0 + 2\delta)$ satisfying

$$0 \leq \zeta \leq 1 \text{ in } [-a, a], \quad \zeta \equiv 1 \text{ in } [x_0 - \delta, x_0 + \delta].$$

Then $h_0 > 0$ in the support of ζ and hence (249) holds for $\alpha \neq -1$, Since, $n > \frac{p}{p-1}$ we can choose $\frac{p-1}{p} - n < \alpha < -1$ such that $\frac{p+1}{\alpha+n+p-1}(\alpha+1) \leq -1$. By (235) we get a contradiction and this proves the result.

(iii): Suppose that the assertion is not true. i.e. $h(t, x_0) = 0$. Since $n \geq 1 + \frac{(p-1)}{p} + \frac{p}{(p-1)}$ we may choose α such that $\frac{p-1}{p} < \alpha < -1$ such that $\frac{p-1}{p}(\alpha+1) \leq -1$ and this yields a contradiction to (249), by uniform Hölder continuity of $h(t, x)$ in x.

(v): We suppose on the contrary that there exists a time $t > 0$, a constant $\delta > 0$ and a smooth function ϕ with support in Ω satisfying

$$h(t, x) > \delta > 0 \text{ for } x \in supp(\phi),$$
$$supp(\phi) \cap supp(h_0) = \emptyset.$$

Let $c > 0$ be a constant. By the dissipation results, we proved in the introduction of this chapter, we deduce that $h_t \in L^2(0, T; H^{-1}((-a, a)))$. Thus, we may choose $\psi = \frac{\phi}{h+c}$ as test function in (251) given by

$$\iint_Q h\psi_t dxdt + \iint_P h^n((h_x^2)^{p/2-1}h_{xx})_x \psi_x dxdt = 0. \tag{251}$$

This gives us

$$\begin{aligned} &\int_\Omega \phi(x)\ln(h(t, x) + c)dx - \int_\Omega \phi(x)\ln(h_0(x) + c)dx \\ = &\iint_{P\cap Q_t} ((h_x^2)^{p/2-1}h_{xx})_x \frac{\phi' h^n}{h+c} dxdt - \iint_{P\cap Q_t} ((h_x^2)^{p/2-1}h_{xx})_x \frac{\phi h_x h^n}{(h+c)^2} dxdt =: L_1 + L_2. \end{aligned} \tag{252}$$

By the choice of ϕ, we know that

$$\int_\Omega \phi(x)\ln(h(x, t) + c)dx - \int_\Omega \phi(x)\ln(h_0(x) + c)dx \to \infty \;\; as\ c \to 0. \tag{253}$$

Since $n \geq 2 + p/(p-1)$, $h^{n/2}((h_x^2)^{p/2-1}h_{xx})_x \in L^2(P \cap Q_t), h_x \in L^2(Q_t)$ and h is bounded in Q_t. Now, to get a contradiction, we will try to bound the last two terms in (252) uniformly. Let $P_t := P \cap Q_t$.

$$|L_1| \leq \left(\iint_{P_t} h^n((h_x^2)^{p/2-1}h_{xx})_x^2 dxdt\right)^{1/2} \left(\iint_{P_t} h^{n-2}(\frac{\phi' h}{h+c})^2\right)^{1/2} \leq C_1. \tag{254}$$

$$|L_2| \leq \left(\iint_{P_t} h^n((h_x^2)^{p/2-1}h_{xx})_x^2 dxdt\right)^{1/2} \left(\iint_{P_t} h^{n-(2+p/(p-1))}(\frac{\phi^2 h_x^2 h^{2+p/(p-1)}}{h+c})^4\right)^{1/2} \leq C_2. \qquad (255)$$

Since C_1 and C_2 are constants independent of c we get a contradiction.

□

3.9 Asymptotic Behavior of Nonnegative Solutions

In this section first we consider nonnegative smooth solutions and prove asymptotic decay and then we do the same thing for nonnegative weak solutions. Some preliminary results proved in the smooth case will also be useful for the weak solution case. The main idea is to control the rate of decrease of the energy functional in terms of itself, which was the same motivation in [93]. For certain reasons we divide this section into two parts.

Asymptotic Behavior of Nonnegative Smooth Solutions

In this section we will use the energy functional $H_p(f)$, defined in (65), to obtain asymptotic behavior of the nonnegative smooth solutions of the equation (67) with initial and boundary conditions (68) and (69).

Note that

$$\frac{d}{dt}H_p[h(t,x)] = -(p-1)^2 \int_\Omega h^n[(h_x^2)^{p/2-1}h_{xx}]_x^2 dx \leq 0.$$

Thus, $H_p(h(t,x))$ is a Lyapunov functional for nonnegative smooth solutions of (67).

Lemma 22 (A zero'th order dissipated energy) : Let h be a nonnegative smooth solution of the problem (67), (68) and (69) with $p \geq 1$. Then

$$t \to \int_\Omega h^{2-n} dx, \quad n \geq 2$$

is non increasing.

Proof: Indeed by differentiating and integrating by parts we have that, for simplicity we write $h = h(t,x)$,

$$\frac{d}{dt}\int_\Omega h^{2-n} dx = (2-n)\int_\Omega h^{1-n} h_t dx$$

$$= -(n-2)(n-1)(p-1)\int_{\Omega}(h_x^2)^{p/2-1}h_{xx}^2dx \le 0,$$

as $n \ge 2$ and $p \ge 1$.

$\square$

Proof of Lemma 5: Note that an integration by parts yields that

$$pH_p(u) = -(p-1)\int_{\Omega} u[(u_x^2)^{p/2-1}u_{xx}]dx.$$

Keeping this in mind we also note that for all $x_0, x \in \Omega$ one has by Cauchy-Schwartz inequality that

$$-\int_{x_0}^{x} u[(u_x^2)^{p/2-1}u_{xx}]_x \le (\int_{\Omega}\frac{u^2}{\psi(u)}dx)^{1/2}(\int_{\Omega}\psi(u)[(u_x^2)^{p/2-1}u_{xx}]_x^2dx)^{1/2}. \tag{256}$$

On the other hand, by integration by parts we have

$$-\int_{x_0}^{x} u[(u_x^2)^{p/2-1}u_{xx}]_xdx = -u(x)[(u_x^2)^{p/2-1}u_{xx}](x) + u(x_0)[(u_x^2)^{p/2-1}u_{xx}](x_0) + \int_{x_0}^{x} u_x(u_x^2)^{p/2-1}u_{xx}dx.$$

Thus, if we denote the right hand side of (256) by $A \in [0,\infty)$ we have, by assuming $u_x(x_0) = 0$

$$A \ge -u(x)(u_x^2)^{p/2-1}u_{xx}(x) + \frac{1}{p}(u_x^2)^{p/2}(x).$$

Now integrating this in x over Ω and applying integration by parts to the first integral we finally deduce that

$$A \ge CH_p(u),$$

where

$$C := \frac{2p-1}{2ap(p-1)}.$$

This immediately gives the result.

$\square$

Proof of Proposition 6:

(i) By the energy dissipation, we deduce that $t \to \|h_x\|_{L^p(\Omega)}$ is non increasing. Moreover, since the mass of h_0 is finite there exists a constant $K = K(p)$ such that

$$\|h(t,x)\|_{L^{\infty}(\Omega)} \le K\|h_{0x}\|_{L^p} =: R_0.$$

Using Lemma 5 with $\psi(h) \equiv h^n$ and $u \equiv h$ we deduce that

$$E[h]D[h] \geq CH_p^2[h],$$

where

$$E[h] := \int_\Omega h^{2-n} dx, \tag{257}$$

$$D[h] := \int_\Omega h^n [(h_x^2)^{p/2-1} h_{xx}]_x^2 dx, \tag{258}$$

and

$$C_1 := \frac{2p-1}{2pa(p-1)}.$$

On the other hand since $0 < n < 2$ we have

$$E[h] \leq R_0^{2-n} \int_\Omega dx = 2aR_0^{2-n}.$$

Thus, we have that

$$D[h] \geq C[H_p(h)]^2,$$

where

$$C := \frac{[\frac{2p-1}{2pa(p-1)}]^2}{2aR_0^{2-n}} p^2.$$

(ii) Note that the above proof works and $C = [\frac{2p-1}{2pa(p-1)}]^2 \frac{p^2}{2a}$.

(iii) By Lemma 5 and Lemma 22 we deduce that

$$D[h] \geq C_2 H_p^2[h],$$

where

$$C_2 := \frac{[\frac{2p-1}{2pa(p-1)}]^2}{\int_\Omega h_0^{2-n} dx}.$$

This easily gives the result. Note that we can get the proof of (*ii*) from here as well.

$\square$

By the Proposition 6 we deduce that

$$H_p[h(t,x)] \leq [H_p[h_0]^{-1} + Ct]^{-1}, t > 0. \tag{259}$$

Hence, from this, $H_p(h(t,x))$ becomes sufficiently small after some finite time and so $h(t,x)$ becomes uniformly bounded from below away from 0. From this point on we can then deduce from linearization that there is an exponential decay.

Asymptotic Behavior of Nonnegative Weak Solutions

We now consider a weak solution of the problem (67), (68) and (69). We assume that

$$\int_\Omega h_0(x)^{2-n} dx < \infty, n > 2 \tag{260}$$

$$\int_\Omega |\log(h_0(x))| dx < \infty, n = 2. \tag{261}$$

and also $\int_\Omega h_0(x)dx =: M < \infty$. These assumptions guarantee the existence of a weak solution.

Recall also the entropy H_ϵ defined by

$$H_\epsilon(h) = \int_\Omega G_\epsilon(h) dx,$$

where

$$G''_\epsilon(h) = \frac{1}{P_\epsilon(h)},$$

and $P_\epsilon(h)$ is given by (72).

Case $0 < n < 1$ **or** $n > 2$

In this case we have that

$$H_\epsilon[h] = \int_\Omega \frac{\epsilon}{c(c-1)} h^c + c_n h^{2-n} dx, \tag{262}$$

where

$$c := 2 - (p + p/(p-1)), \tag{263}$$

and

$$c_n := \frac{1}{(n-1)(n-2)}. \tag{264}$$

Note also that both of the terms appearing in (262) are positive and we have, by the dissipation of the entropy H_ϵ, that

$$H_\epsilon[h_{0\epsilon}] \geq \begin{cases} \frac{1}{c(c-1)} \int_\Omega [\epsilon h^c + h^{2-n}] dx & \text{if } \frac{1}{c(c-1)} > c_n, \\ c_n \int_\Omega [\epsilon h^c + h^{2-n}] dx & \text{if } c_n > \frac{1}{c(c-1)}. \end{cases}$$

Note also that it is not difficult to show

$$H_\epsilon[h_{0\epsilon}] \leq \int_\Omega \left[C_p \epsilon^{(1-c)\theta} + c_n h_0^{2-n} \right] dx,$$

where c and c_n are given by (263) and (264) respectively and C_p is a finite constant depending on p. This clearly gives a uniform upper bound on $H_\epsilon[h_\epsilon]$ as $\epsilon \searrow 0$.

Proof of Proposition 7 : Given $t > 0$, we first note that it is not difficult to show that $H_\epsilon[h_{0\epsilon}]$ is bounded from above and from below uniformly in ϵ as $\epsilon \searrow 0$. Note also that

$$H_p[h_{0\epsilon}] = H_p[h_0],$$

$$H_\epsilon[h_{0\epsilon}] \to c_n E[h_0],$$

where c_n and E are given by (263) and (257) respectively. Applying the Lemma 5 with $u \equiv h_\epsilon(t,x)$ and $\psi \equiv P_\epsilon(h_\epsilon(t,x))$ and noticing that

$$\frac{h_\epsilon^2}{P_\epsilon(h_\epsilon)} = \frac{\epsilon}{h_\epsilon^c} + h_\epsilon^{2-n},$$

where c is given by (263), we deduce that

$$c_{pn} H_\epsilon[h_{0\epsilon}] D_\epsilon[h_\epsilon] \geq K_p H_p^2[h_\epsilon(t,x)], \tag{265}$$

where c_{pn}, K_p are positive constants which can be determined explicitly(we leave this for the reader), and

$$D_\epsilon[h_\epsilon] := \int_\Omega P_\epsilon(h_\epsilon(t,x)) \{[(h_{\epsilon x}^2)^{p/2-1} h_{\epsilon xx}]_x\}^2 dx. \tag{266}$$

On the other hand, we have

$$\frac{d}{dt} H_p[h_\epsilon(t,x)] = -(p-1) D_\epsilon[h_\epsilon(t,x)], \tag{267}$$

where D_ϵ is given by (266). Combination of (265) and (267) and letting $\epsilon \to 0$ finishes the proof.

□

Clearly the Proposition 7 yields that

$$H_p[h(t,x)] \le H_p[h_0]\left(1 + \tau_1 H_p[h_0]t\right)^{-1}, \tau_1 > 0. \tag{268}$$

This implies that whenever $H_p[h(t,x)]$ is small enough $h(t,x)$ becomes bounded below away from 0, and after this point on we have exponential decay by linearization.

Remaining case for n :

Now we consider the remaining case. First let us introduce the notation. For $t \ge 0$ we define

$$J_\epsilon(t) := \frac{1}{p}\int_\Omega |h_{\epsilon x}|^p dx,$$

and

$$I_\epsilon(t) := (p-1)\int_\Omega P_\epsilon\left[(h_{\epsilon x}^2)^{p/2-1}h_{\epsilon xx}\right]_x^2 dx,$$

where P_ϵ is given in (72). It is easy to see that

$$-\int_\Omega P_\epsilon^{1/2}\left[(h_{\epsilon x}^2)^{p/2-1}h_{\epsilon xx}\right]_x h_{\epsilon x}dx \le (CI_\epsilon J_\epsilon)^{1/2},$$

where C is a positive constant depending on $p, |\Omega|, \|h_{0x}\|_{L^p(\Omega)}$. We also use the following notation $g_\epsilon(s) := P_\epsilon^{1/2}(s)$. For future reference we compute

$$g_\epsilon(s) = s^{(2+p/(p-1))/2}\left(\epsilon + s^{2+p/(p-1)-n}\right)^{-1/2}$$

and from this one easily gets

$$g_\epsilon''(s) = (\epsilon + s^m)^{-5/2}\left[C_1\epsilon s^{\alpha+m-2} + C_2\epsilon^2 s^{\alpha-2} + C_3 s^{2m+\alpha-2}\right],$$

where

$$\alpha := \frac{p}{2(p-1)} + 1, \qquad m := 2 + \frac{p}{p-1} - n, \tag{269}$$

$$C_1 := (-\frac{3}{2}\alpha m + \alpha(\alpha-1) + (\alpha - \frac{1}{2}m)(\alpha + m - 1)),$$

$$C_2 := \alpha(\alpha-1),$$

$$C_3 := (\alpha - \frac{1}{2}m)(\alpha + m - 1) - \frac{3}{2}m(\alpha - \frac{1}{2}m).$$

Note that for $n \in [1,2]$

$$\frac{p}{(p-1)} \leq m \leq 1 + \frac{p}{(p-1)} < 2 + \frac{p}{(p-1)},$$

which is the analogous inequality for our case. Note that this actually works for $0 < n < 2 + \frac{p}{(p-1)}$. It is not difficult to see that the coefficients C_1 and C_3 are non negative and hence we can eliminate the terms involving C_1 and C_3. This leads to

$$\begin{aligned}(CI_\epsilon J_\epsilon)^{1/2} &\geq \int_\Omega g_\epsilon(h_\epsilon)(h_{\epsilon x}^2)^{p/2-1} h_{\epsilon xx}^2 dx \\ &- \epsilon^2 \frac{\alpha(\alpha-1)}{(p+1)} \int_\Omega (\epsilon + h_\epsilon^m)^{-5/2} h_\epsilon^{\alpha-2} (h_{\epsilon x}^2)^{p/2-1} h_{\epsilon x}^4 dx \\ &=: I_1 - \frac{1}{(p+1)} C_2 \epsilon^2 I_{2,2}. \end{aligned} \tag{270}$$

It is also not difficult to obtain an upper bound for the L^∞ norm of $h(t,.)$ uniformly in t. Let M_ϵ be the upper bound for $|h_\epsilon|_{L^\infty(\Omega)}$. Also we can easily deduce that

$$g_\epsilon(h_\epsilon) \geq (\epsilon + M_\epsilon^m)^{-1/2} (\epsilon + M_\epsilon)^{-(p-2)/(2(p-1))} h_\epsilon^2.$$

Using this we obtain that

$$\begin{aligned}(CI_\epsilon J_\epsilon)^{1/2} &\geq (\epsilon + M_\epsilon^m)^{-1/2} (\epsilon + M_\epsilon)^{-(p-2)/(2(p-1))} \int_\Omega h_\epsilon^2 (h_{\epsilon x}^2)^{p/2-1} h_{\epsilon xx}^2 dx \\ &- \frac{1}{(p+1)} C_2 \epsilon^2 I_{2,2} =: L_\epsilon J_1 - \frac{1}{(p+1)} C_2 \epsilon^2 I_{2,2}. \end{aligned} \tag{271}$$

For the second term in (271), using exactly the same idea as in Lemma 2 of [93], we obtain.

$$\begin{aligned}(CI_\epsilon J_\epsilon)^{1/2} &\geq L_\epsilon J_1 - S_\epsilon \epsilon^w \int_\Omega h_\epsilon^{-2} (h_{\epsilon x}^2)^{p/2-1} h_{\epsilon x}^4 dx \\ &= L_\epsilon J_1 - S_\epsilon \epsilon^w J_2. \end{aligned} \tag{272}$$

We note that the constants can be determined explicitly and moreover S_ϵ is a finite constant with $S_\epsilon \to S$ as $\epsilon \to 0$ and S is finite. On the other hand, $L_\epsilon \to L$, where L is finite constant, as $\epsilon \to 0$.

To proceed further we need to use Lemma 11 and prove the following Lemma that is analogous to Lemma 4 of [93].

Lemma 23 : Let $v := h_\epsilon$ be the solution of (73). Then there exists constants C and α such that

$$\int_\Omega v^2 (v_x^2)^{p/2-1} v_{xx}^2 dx \geq C r^\alpha \int_\Omega |v_x|^p dx, \tag{273}$$

where $r := \int_\Omega h_\epsilon dx > 0$.

Proof: Note that an integration by parts yields

$$\int_\Omega |v_x|^p dx = -(p-3)\int_\Omega v(v_x^2)^{p/2-1} v_{xx} dx.$$

Note also that by Cauchy-Schwarz inequality

$$J_\epsilon(h_\epsilon) \le C(M_\epsilon, m)\left(\int_\Omega h_\epsilon^{1+p/(2(p-1))}(h_{\epsilon x}^2)^{p/2-1} h_{\epsilon xx}^2 dx\right)^{1/2} J_\epsilon(h_\epsilon)^{(p-2)/p}|\Omega|^{4/p}.$$

Letting $J_1^* := \int_\Omega h_\epsilon^{1+p/(2(p-1))}(h_{\epsilon x}^2)^{p/2-1} h_{\epsilon xx}^2 dx$ we get

$$J_1^* \ge A(M_\epsilon, m, p, |\Omega|)J_\epsilon^{4/p}, \tag{274}$$

where A is a finite constant. For any $0 < \lambda < (\frac{r}{|\Omega|C_1})^p$, where C_1 is a constant given below, we have $J_1^* \ge C\lambda^{4/p-1}(\int_\Omega |h_{\epsilon x}|^p dx)$, if $\int_\Omega |h_{\epsilon x}|^p dx \ge \lambda$. On the other hand, by using Sobolev and Poincare inequalities we also have

$$J_1^* \ge C(\frac{r}{|\Omega|} - C_1\lambda^{1/p})^p(\int_\Omega |h_{\epsilon x}|^p dx),$$

whenever $\int_\Omega |h_{\epsilon x}|^p dx \le \lambda$. It follows, as in [93], that

$$J_1^* \ge \min\{C\lambda^{4/p-1}, C'(\frac{r}{|\Omega|} - C_1\lambda^{1/p})^p\}\int_\Omega |h_{\epsilon x}|^p dx$$

for $0 < \lambda < (\frac{r}{|\Omega|C_1})^p$. It follows that

$$J_1^* \ge CJ_\epsilon, \tag{275}$$

where the finite constant C can be obtained explicitly, and it depends on $m, p, |\Omega|, \int_\Omega h_\epsilon dx$ and constants of the Sobolev and Poincare inequalities. □

Combining what we have so far we deduce that

$$C_\epsilon J_\epsilon - (CI_\epsilon J_\epsilon)^{1/2} \le C_p C_\epsilon \epsilon^w \int_\Omega (h_{\epsilon x}^2)^{p/2-1} h_{\epsilon xx}^2 dx, \tag{276}$$

from which we obtain

$$C_{p\epsilon}^2 J_\epsilon \le I_\epsilon + C_p C_\epsilon \epsilon^w \int_\Omega (h_{\epsilon x}^2)^{p/2-1} h_{\epsilon xx}^2 dx. \tag{277}$$

Noting that $I_\epsilon = \frac{d}{dt}J_\epsilon$, we obtain from (277)

$$\frac{d}{dt}J_\epsilon \leq -C_p^* C_\epsilon^2 J_\epsilon + C_p C_\epsilon \epsilon^w \int_\Omega (h_{\epsilon x}^2)^{p/2-1} h_{\epsilon xx}^2 dx, \tag{278}$$

where $C_p^*, C_\epsilon, C_p, w$ are all positive. Applying a version of the Gronwall's inequality we deduce that

$$\begin{aligned} J_\epsilon(t) &\leq e^{-C_p C_\epsilon^2 t}[J_\epsilon(0) + C_p C_\epsilon \epsilon^w \int_0^t \int_\Omega (h_{\epsilon x}^2)^{p/2-1} h_{\epsilon xx}^2 dxdt] \\ &\leq e^{-C_p C_\epsilon^2 t} J_\epsilon(0) + C_p C_\epsilon \epsilon^w \int_0^t \int_\Omega (h_{\epsilon x}^2)^{p/2-1} h_{\epsilon xx}^2 dxdt. \end{aligned} \tag{279}$$

Noting that $w > 0$, $\int_0^t \int_\Omega (h_{\epsilon x}^2)^{p/2-1} h_{\epsilon xx}^2 dxdt < \infty$ by the energy dissipation and $C_\epsilon \to C_0 < \infty$ as $\epsilon \to 0$ and

$$\int_\Omega |h_x|^p dx \leq \liminf_{\epsilon \searrow 0} \int_\Omega |h_{\epsilon x}|^p dx, \forall t > 0,$$

we pass to the limit as $\epsilon \searrow 0$ and deduce finally that

$$J[h(t,.)] \leq J[h_0(\cdot)] \exp(-Ct),$$

where C is a finite positive constant.

$\square$

Hence we proved the exponential decay directly in this case.

Remark : The approach of [93, 95] can be employed to similar equations. Consider the so called "modified thin film equation" [11, 9, 18] given by

$$u_t = -u^n u_{xxxx}, \tag{280}$$

under periodic or no-flux boundary conditions. Here if one considers the energy functional $E[u(t,x)] := \int_\Omega u_{xx}^2 dx$ then the following dissipation result holds for positive smooth solutions of (280).

$$\frac{d}{dt}\int_\Omega u_{xx}^2 dx = -\int_\Omega u^n u_{xxxx}^2 dx.$$

On the other hand we also have by Cauchy-Schwarz inequality that

$$\int_\Omega u u_{xxxx} dx \leq \left(\int_\Omega u^n u_{xxxx}^2\right)^{1/2} \left(\int_\Omega u^{2-n}\right)^{1/2}.$$

This yields after integration by parts, and using the boundary conditions that

$$D[u(t,x)] := \int_\Omega u^n u_{xxxx}^2 dx \geq \frac{E^2[u(t,x)]}{\int_\Omega u^{2-n} dx}. \tag{281}$$

Thus, if $n \leq 2$ then we have

$$D[u(t,x)] \geq CE^2[u(t,x)].$$

Finally, we obtain that

$$E[u(t,x)] \leq \frac{E[u_0]}{1 + CE[u_0]t}, \tag{282}$$

where $u_0(x) = u(t=0,x)$. Note that (282) gives a polynomial decay of positive smooth solutions of (280).

CHAPTER IV

DETAILS OF THE RESULTS ON THE CAUCHY PROBLEM (95)

4.1 Introduction

In the remaining part of the thesis we study the asymptotic behavior of smooth solutions $h(t,x)$ to the thin film equation

$$h_t = -(hh_{xxx})_x, \qquad x \in \mathbb{R}, t > 0, \tag{283}$$

with

$$h(0,x) = h_0(x) \geq 0, \qquad x \in \mathbb{R}. \tag{284}$$

Equation (283) is a special case of the so-called thin film equation

$$h_t = -(h^n h_{xxx})_x, \qquad x \in \mathbb{R}, t > 0, \tag{285}$$

for $n > 0$. (285) has been derived from a lubrication approximation to model the surface tension dominated motion of viscous liquid films and spreading droplets [12, 19, 72].

We show that for a fairly general class of initial data, the classical solutions of (283) converge toward certain self–similar solutions in the $H^1(\mathbb{R})$ norm. We also estimate the rate of convergence. Previous work [27] had established this convergence in the $L^1(\mathbb{R})$, and while these authors raised the question of $H^1(\mathbb{R})$ convergence, which is natural for the equation, their methods did not address the issue.

In what follows here, we make use of functionals involving higher order derivatives, and to justify the calculations we make, we must assume that the solutions with which we work are classical. This is in contrast to the work in [27], where strong solutions were treated. The results in [17], where the issue of finite time blow up of solutions for the thin film type equations has been discussed, show that in general it is possible for classical solutions to break down in finite time. However, the equipartition mechanism that we introduce here provides a new perspective on equilibriation, and it may well be possible to establish it for a more general class of solutions.

The equation (283) is gradient flow for the energy $E_0(u)$ where

$$E_0(h) = \frac{1}{2}\int_{\mathbb{R}} h_x^2(x)\mathrm{d}x\ ,$$

in that

$$h_t = \left(h\left(\frac{\delta E_0(u)}{\delta h}\right)_x\right)_x\ .$$

This has the consequence that for solutions $h(t,\cdot)$, $E_0(h(t,\cdot))$ is monotone decreasing in time. Also, since the equation is also a conservation law, the total mass

$$M = \int_{\mathbb{R}} h(t,x)\mathrm{d}x$$

is conserved.

Moreover, the equation (283) has a scale invariance, and self–similar solutions. If one introduces

$$v(t,x) = \alpha(t)h(\beta(t),\alpha(t)x), \tag{286}$$

where

$$\alpha(t) = e^t \quad \text{and} \quad \beta(t) = \frac{e^{5t}-1}{5}\ , \tag{287}$$

it becomes

$$v_t = (xv - vv_{xxx})_x\ , \qquad x \in \mathbb{R},\ t > 0, \tag{288}$$

$$v(0,x) = v_0(x), \qquad x \in \mathbb{R}. \tag{289}$$

The equation (288) has a unique steady state, found by Smythe and Hill [91]:

$$v^{(\infty)}(x) = \frac{1}{24}\left(C^2 - x^2\right)_+^2\ , \tag{290}$$

where g_+ indicates the positive part of g, and where the constant $C = C(M)$ is determined by the requirement that $\int_{\mathbb{R}} v^{(\infty)}(x)\mathrm{d}x = \int_{\mathbb{R}} h_0(x)\mathrm{d}x$. Source type solutions of the thin film equation (285) has been studied in [10] and the uniqueness of the steady states of the rescaled equation in the general case is derived from the uniqueness of source type solutions $U(t,x)$ for (285), requiring $U_x(t,x) = 0$ at the edge o the support.

Clearly, if a solution $v(t,x)$ of (288) approaches $v^{(\infty)}$, the corresponding solution $h(t,x)$ of (283) approaches to the corresponding self–similar solution. For the investigation of the rates at which this takes place, it is important that (288) also describes a gradient flow: Introduce the energy functional $E(v)$ where

$$E(v) = \frac{1}{2}\int_{\mathbb{R}} (v_x^2(x) + x^2 v(x))\mathrm{d}x \ .$$

Then, (288) can be rewritten as

$$v_t = \left(v\left(\frac{\delta E(v)}{\delta v}\right)_x\right)_x \ .$$

Clearly then, for any solution $v(t,x)$ of (288), $E(v(t,\cdot))$ is non increasing in t. Define

$$E(v|v^{(\infty)}) = \frac{1}{2}\int_{\mathbb{R}} |v_x - v_x^{(\infty)}|^2 \mathrm{d}x \ ,$$

where $v^{(\infty)}$ is the stationary solution with the same mass as v. Our goal is to estimate the rate of at which $E(v(t,\cdot)|v^{(\infty)})$ converges to zero. Indeed, our analysis will provide the first proof that for general initial conditions this convergence does indeed take place. Note that this convergence is exactly the convergence of $v(t,\cdot)$ to $v^{(\infty)}$ in the $H^1(\mathbb{R})$ norm.

By using the explicit formula for the function $v^{(\infty)}$ and proceeding as in the analysis of second-order degenerate diffusion in [28] we obtain

$$\begin{aligned} E(v|v^{(\infty)}) &= \frac{1}{2}\int_{\mathbb{R}} |v_x - v_x^{(\infty)}|^2 \mathrm{d}x \\ &= E(v) - E(v^{(\infty)}) - \int_{\{v^{(\infty)}=0\}} \frac{x^2}{2} v \mathrm{d}x - \frac{C^2}{6}\int_{\{v^{(\infty)}=0\}} v \mathrm{d}x \\ &\le E(v) - E(v^{(\infty)}), \end{aligned} \tag{291}$$

where C is the constant appearing in the definition of $v^{(\infty)}$.

To estimate the rate of convergence in $H^1(\mathbb{R})$, it therefore suffices to prove that the *excess energy*, $E(v) - E(v_\infty)$, decreases to zero. Toward this end we define the *energy dissipation*, $D_E(v)$, given by

$$D_E(v(t,\cdot)) = -\frac{\mathrm{d}}{\mathrm{d}t}(E(v) - E(v_\infty)) = -\frac{\mathrm{d}}{\mathrm{d}t}E(v) \ .$$

It follows from (288) that $D_E(v)$ is given by

$$D_E(v) := \int_{\mathbb{R}} v\,(v_{xxx} - x)^2\,dx \tag{292}$$

Our object here is to prove a lower bound on $D_E(v(t,\cdot))$ in terms of $E(v(t,\cdot)|v^{(\infty)})$ which we shall use to prove that for a broad class of initial data, $\lim_{t\to\infty} E(v(t,\cdot)|v^{(\infty)}) = 0$, and to estimate the rate at which this convergence takes place.

In obtaining our energy dissipation bound, we shall make crucial use of an entropy dissipation bound. Indeed, as shown in [27], the equation (288) can be written as

$$v_t = -\left(\Phi(v)\left[\frac{x^2}{2} + h(v)\right]_{xx}\right)_{xx} + \left(v\left[\frac{x^2}{2} + h(v)\right]_x\right)_x ,$$

with $h(v) = \sqrt{6}v^{1/2}$ and $\Phi(v) = vh'(v)$. This leads to the exact form of the entropy associated to the unique steady state $v^{(\infty)}$, given in (290), which is

$$H(v) = \int_{\mathbb{R}} \left(\frac{x^2}{2}v(x) + 2\sqrt{\frac{2}{3}}v^{3/2}(x)\right) \mathrm{d}x .$$

One defines the relative entropy by

$$H(v|v^{(\infty)}) = H(v) - H(v^{(\infty)}) .$$

As one can check, $v^{(\infty)}$ minimizes H for given total mass. This relative entropy had already been investigated earlier in the context of a second order evolution equation, namely a special case of the porous medium equation for which $v^{(\infty)}$ is also a stationary solution. In fact, the porous medium equation in question is simply the gradient flow for $H(v)$ in the same way that (288) is gradient flow for the energy E. A truly remarkable discovery [27] of Carrillo and Toscani is that $H(v(t,\cdot))$ is also monotone decreasing for solutions of (288), despite the fact that this equation is gradient flow for the entropy and not the energy. Indeed, Carrillo and Toscani have proved that

$$\frac{\mathrm{d}}{\mathrm{d}t}H(v(t,\cdot)|v^{(\infty)}) \leq -D_H(v(t,\cdot))$$

where the *partial entropy dissipation* $D_H(v)$ is given by

$$D_H(v) := \int_{\mathbb{R}} v\left(\frac{x^2}{2} + \sqrt{6}v^{1/2}\right)_x^2 \mathrm{d}x . \tag{293}$$

We use the term *partial* since full entropy dissipation is the sum of two positive terms, one of which is D_H. Interestingly enough, D_H is the exact entropy dissipation for $H(v|v^{(\infty)})$ for solutions of a porous medium equation. Moreover, as was already established in the investigation of the porous medium equation, one has the entropy–entropy dissipation bound

$$H(v|v^{(\infty)}) \leq \frac{1}{2}D_H(v) . \tag{294}$$

This has the consequence that for solution of (288)

$$H(v(t,\cdot)|v^{(\infty)}) \leq e^{-2t} H(v_0|v^{(\infty)}) \ .$$

Unfortunately, D_E is a much more complicated functional than D_H, and we do not possess a bound of this simple type relating $E(v|v^{(\infty)})$ and $D_E(v)$, and it is not even clear at this point that $E(v(t,\cdot)|v^{(\infty)})$ will generally tend to zero at all.

We shall show here that this convergence does occur, and estimate the rate, using an *equipartition theorem* for solutions of (288).

To explain, consider any smooth solution of (288) with finite energy $E(v(t,\cdot))$. Define

$$\alpha(v) = \frac{1}{2}\int_{\mathbb{R}} x^2 v(x)\mathrm{d}x \qquad \text{and} \qquad \beta(v) = \frac{1}{2}\int_{\mathbb{R}} v_x^2(x)\mathrm{d}x$$

so that

$$E(v) = \alpha(v) + \beta(v) \ .$$

By a simple computation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha(v(t,\cdot)) = -2\alpha(v(t,\cdot)) + 3\beta(v(t,\cdot)) \ .$$

It follows that

$$2\alpha(v^{(\infty)}) = 3\beta(v^{(\infty)}) \ . \tag{295}$$

Analogously to the way to we defined relative entropies and energies, we define $\alpha(v|v^{(\infty)})$ and $\beta(v|v^{(\infty)})$ respectively by

$$\alpha(v|v^{(\infty)}) = \alpha(v(t,\cdot)) - \alpha(v^{(\infty)}) \qquad \beta(v|v^{(\infty)}) = \beta(v(t,\cdot)) - \beta(v^{(\infty)}).$$

Then, by (295),

$$2\alpha(v|v^{(\infty)}) - 3\beta(v|v^{(\infty)}) = 2\alpha(v) - 3\beta(v) \ . \tag{296}$$

We shall prove here that for a general class of classical solutions to (288),

$$\lim_{t\to\infty}(2\alpha(v(t,\cdot)) - 3\beta(v(t,\cdot))) = 0 \ . \tag{297}$$

We refer to this as *asymptotic equipartition* for solutions of (288).

To employ this, we use the entropic convergence result of Carrillo and Toscani to show that furthermore,

$$\lim_{t\to\infty}(\alpha(v(t,\cdot)) - \alpha(v^{(\infty)})) = 0 \, . \tag{298}$$

Combining (295), (297) and (298), we then have that

$$\lim_{t\to\infty}(\beta(v(t,\cdot)) - \beta(v^{(\infty)}) = 0 \, . \tag{299}$$

Combining ((298) and (299), we then have that $\lim_{t\to\infty} E(v(t,\cdot)|v^{(\infty)}) = 0$. In proving all of this we shall keep track of the rate, so that our final result is quantitative.

The key to all of this is an identity expressing $2\alpha(v) - 3\beta(v)$ in terms of a iterated integrals. Suppose that v is a nonnegative smooth function, then as we shall see,

$$2\alpha(v) - 3\beta(v) = 2\int_{-\infty}^{0}\left(\int_{-\infty}^{x} v(z)(v_{zzz}(z) - z)\mathrm{d}z\right)\mathrm{d}x \; - 2\int_{0}^{\infty}\left(\int_{x}^{\infty} v(z)(v_{zzz}(z) - z)\mathrm{d}z\right)\mathrm{d}x \, . \tag{300}$$

Comparing this with (292), one see the possibility of estimating $2\alpha(v)-3\beta(v)$ in terms of something involving $D_E(v)$. In fact, we shall show, under the condition the fourth moment of the initial data is finite, that there is a finite constant K so that

$$|2\alpha(v) - 3\beta(v)| \leq K(D_E(v))^{1/2} \, . \tag{301}$$

This shall be enough to deduce (298). Here is our main result.

Theorem 24 (The Main Theorem) :

For all classical solutions $v(t,x)$ of (288) with smooth, non negative initial data v_0 such that $M_0(v_0)$, $M_4(v_0)$ and $E(v_0)$ are all finite, there is a finite constant C, depending only on $M_0(v_0)$, $M_4(v_0)$ and $E(v_0)$, such that for all $t > 0$, $v(x,t)$ satisfies

$$E(v(t,\cdot)|v^{(\infty)}) \leq \frac{C}{\sqrt{t}} \, . \tag{302}$$

As noted above, $E(v(t,\cdot)|v^{(\infty)}) = \frac{1}{2}\int_{\mathbb{R}} |v_x(t,\cdot) - v_x^{(\infty)}|^2\mathrm{d}x$, so this explicitly estimates the rate of convergence of $v(t,\cdot)$ to $v^{(\infty)}$ in the $H^1(\mathbb{R})$ norm. The fact that this is only power law decay reflects the fact that our proof only gives a power law decay on the rate of equipartition. If one could show the equipartition to take place exponentially fast, then one would get exponential convergence in Theorem 24. But we do not, at present, know whether the equipartition will in general take place exponentially fast. However, it is possible to get a slightly better rate with the present methods: As we shall explain following the proof of Theorem 24, one can improve the right hand side to $C_\epsilon/T^{1-\epsilon}$ for any $\epsilon > 0$.

4.2 Some *a priori* bounds

The main result in the section is the moment bound in Lemma 27. Its proof requires some simpler bounds which we give in the first two lemmas.

Lemma 25 : Any integrable non negative function v on $\mathbb{R}$ for which $E(v) < \infty$ is bounded. More precisely,

$$\|v\|_\infty \le M + \left(\int_{\mathbb{R}} v_x^2(x)\mathrm{d}x\right)^{1/2} \le M + (E(v))^{1/2}$$

where M is the total mass $\int_{\mathbb{R}} v(x)\mathrm{d}x$.

Proof: For any x_0, the average value of v over the interval $[x_0, x_0+1]$ is no greater than M since the length of the interval is 1 and

$$\int_{[x_0,x_0+1]} v(x)\mathrm{d}x \le M\ .$$

Hence there is a point $y_0 \in [x_0, x_0+1]$ such that $v(y_0) \le M$. But then

$$v(x_0) = v(y_0) - \int_{x_0}^{y_0} v_x(x)\mathrm{d}x \le M + \left(\int_{\mathbb{R}} v_x^2(x)\mathrm{d}x\right)^{1/2}\ .$$

□

Lemma 26 : Any integrable non negative function v on $\mathbb{R}$ for which $E(v) < \infty$ and $D_E(v) < \infty$ satisfies

$$\int_{\mathbb{R}} v^{-3/2} v_x^4 \mathrm{d}x \ \le 2D_E(v) + 36H(v)\ .$$

Proof: By the Minkowskii inequality in $L^2(\mathbb{R}, v(x)\mathrm{d}x)$,

$$\left(\int_{\mathbb{R}} v(v_{xxx})^2\mathrm{d}x\right)^{\frac{1}{2}} = \left(\int_{\mathbb{R}} v((v_{xxx}-x)+x)^2\mathrm{d}x\right)^{1/2} \le (D_E(v))^{1/2} + \left(\int_{\mathbb{R}} vx^2\mathrm{d}x\right)^{1/2}\ .$$

Now, for $A > 0$ consider the following inequality:

$$\int_{\mathbb{R}} \left(v^{1/2}v_{xxx} + Av_x\right)^2 dx \ge 0.$$

Integrating this by parts, we deduce that

$$\int_{\mathbb{R}} v(v_{xxx})^2 \mathrm{d}x + A^2 \int_{\mathbb{R}} v_x^2 \mathrm{d}x \geq \frac{A}{6} \int_{\mathbb{R}} v^{-3/2} v_x^4 dx + 2A \int_{\mathbb{R}} v^{1/2} v_{xx}^2 dx. \qquad (303)$$

Choosing $A = 6$, we deduce the result. $\square$

We shall need certain moment bounds. For future use, let us define for all positive integers k,

$$M_k(v) = \int_{\mathbb{R}} x^k v(x) \mathrm{d}x \ .$$

Since our goal is to show that a solution $v(t, x)$ of (288) is tending towards a functions of compact support, namely $v^{(\infty)}$, one would expect to be able to show that $M_k(v(t, \cdot))$ stays bounded uniformly in t for all k. For $k = 2$, this is obvious since $E(v) \geq M_2(v)$, and $E(v(t, \cdot))$ is non increasing. Our analysis shall require a bound in $M_4(v(t, \cdot))$.

Theorem 27 : Let $v(t, x)$ be any classical solution of (288) for which the initial data v_0 is integrable and non negative, and satisfies $M_4(v_0) < \infty$ and $E(v_0) < \infty$. Then

$$M_4(v(t, \cdot)) \leq 2D_E(v) + 36E(v) \ .$$

Proof: We first compute

$$\begin{aligned} \frac{d}{dt} M_4(v(t, \cdot)) &= \frac{d}{dt} \int_{\mathbb{R}} x^4 v(t, x) dx = \int_{\mathbb{R}} x^4 (xv - vv_{xxx})_x dx \\ &= -4 \int_{\mathbb{R}} x^4 v dx - 4 \int_{\mathbb{R}} v^2 dx + 18 \int_{\mathbb{R}} x^2 v_x^2 dx. \end{aligned} \qquad (304)$$

The next to last term on the right can be discarded, but the last term requires further analysis. Using Lemmas 25 and 26, and the Cauchy-Schwartz inequality, we deduce that

$$\begin{aligned} \int_{\mathbb{R}} x^2 v_x^2 dx &= \int_{\mathbb{R}} x^2 v^{3/4} v^{-3/4} v_x^2 dx \\ &\leq \left(\int_{\mathbb{R}} x^4 v^{3/2} dx \right)^{1/2} \left(\int_{\mathbb{R}} \frac{v_x^4}{v^{3/2}} dx \right)^{1/2} \\ &\leq C_1 \left(C_2 + D_E(v(t, \cdot)) \right)^{1/2} \left(M_4(v(t, \cdot)) \right)^{1/2}, \end{aligned} \qquad (305)$$

where C_1 and C_2 are constants depending only on $E(v_0)$ and the total mass of v_0. Now, define

$$\phi(t) = (M_4(v(t, \cdot)))^{1/2} \quad \text{and} \quad f(t) = 18 C_1 \left(C_2 + D_E(v(t, \cdot)) \right)^{1/2} \ .$$

Then we deduce from (304) and (305) that

$$\frac{d}{dt}\phi(t) \le -4\phi(t) + f(t) \ .$$

Therefore,

$$\phi(t) \le \phi(0) + e^{-4t}\int_0^t e^{4s} f(s)\mathrm{d}s \ .$$

Note that $f(t) \le f_1(t) + f_2(t)$ where

$$f_1(t) = 18C_1(C_2)^{1/2} \qquad \text{and} \qquad f_2(t) = 18C_1\left(D(v(t,\cdot))\right)^{1/2} \ .$$

Note that f_1 is bounded on $\mathbb{R}_+$, and f_2 is square integrable on $\mathbb{R}_+$:

$$\int_0^\infty f_2^2(t)\mathrm{d}t \ = (18C_1)^2\int_0^\infty D(v(t,\cdot))\mathrm{d}t \ \le (18C_1)^2 H(v_0) \ .$$

But clearly,

$$e^{-4t}\int_0^t e^{4s} f_1(s)\mathrm{d}s \le \|f_1\|_\infty e^{-4t}\int_0^t e^{4s}\mathrm{d}s \le \frac{\|f_1\|_\infty}{4} \ ,$$

and

$$e^{-4t}\int_0^t e^{4s} f_1(s)\mathrm{d}s \le e^{-4t}\left(\frac{e^{8t}-1}{8}\right)^{1/2}\|f\|_2 \le \frac{\|f\|_2}{\sqrt{8}} \ .$$

Hence we have

$$\phi(t) \le \phi(0) + \frac{\|f_1\|_\infty}{4} + \frac{\|f\|_2}{\sqrt{8}}$$

uniformly in t. The right hand side is a constant depending only on $M_4(v_0)$, $E(v_0)$, and the total mass of v_0, $M_0(v_0)$. □

4.3 The iterated integral identity

The key to result in this section is an identity for $2\alpha(v) - 3\beta(v)$ in terms of iterated integrals, where the integrand is related the the integrand in $D_E(v)$.

Lemma 28: For any smooth function v that vanishes at $\pm\infty$

$$2\alpha(v) - 3\beta(v) = 2\int_{-\infty}^0\left(\int_{-\infty}^x v(z)(v_{zzz}(z) - z)\mathrm{d}z\right)\mathrm{d}x \ - 2\int_0^\infty\left(\int_x^\infty v(z)(v_{zzz}(z) - z)\mathrm{d}z\right)\mathrm{d}x \ . \quad (306)$$

Proof: We first compute

$$J_1 := \int_{-\infty}^0\left(\int_{-\infty}^x v(z)v_{zzz}(z)\mathrm{d}z\right)\mathrm{d}x \ - \int_0^\infty\left(\int_x^\infty v(z)v_{zzz}(z)\mathrm{d}z\right)\mathrm{d}x \ .$$

Integrating by parts in the inner integrals, we obtain respectively that

$$\begin{aligned}
\int_{-\infty}^{x} v(z)v_{zzz}(z)\mathrm{d}z &= v(x)v_{xx}(x) - \int_{-\infty}^{x} v_z(z)v_{zz}(z)\mathrm{d}z \\
&= v(x)v_{xx}(x) - \int_{-\infty}^{x} (v_z(z)/2)_z^2\mathrm{d}z \\
&= v(x)v_{xx}(x) - (v_x(x)/2)^2
\end{aligned} \tag{307}$$

$$\begin{aligned}
\int_{x}^{\infty} v(z)v_{zzz}(z)\mathrm{d}z &= -v(x)v_{xx}(x) - \int_{x}^{\infty} v_z(z)v_{zz}(z)\mathrm{d}z \\
&= -v(x)v_{xx}(x) - \int_{x}^{\infty} (v_z(z)/2)_z^2\mathrm{d}z \\
&= -v(x)v_{xx}(x) + (v_x(x)/2)^2
\end{aligned} \tag{308}$$

Therefore, integrating by parts once more,

$$J_1 = -\frac{3}{2}\int_{-\infty}^{+\infty} v_x^2(x)\mathrm{d}x \ .$$

Next, we compute

$$J_2 := -\int_{-\infty}^{0}\left(\int_{-\infty}^{x} v(z)z\mathrm{d}z\right)\mathrm{d}x \ + \int_{0}^{\infty}\left(\int_{x}^{\infty} v(z)z\mathrm{d}z\right)\mathrm{d}x \ .$$

Changing the order of integration we easily find

$$J_2 = \int_{-\infty}^{+\infty} x^2 v(x)\mathrm{d}x \ .$$

Combining the pieces, the identity is proved. □

Lemma 29 : For any smooth, non negative v such that $M_0(v)$, $M_4(v)$ and $E(v)$ are all finite, there is a finite constant K, depending only on $M_0(v)$, $M_4(v)$ and $E(v)$, such that

$$|2\alpha(v) - 3\beta(v)| \leq K\left(D_E(v)\right)^{1/2} \ . \tag{309}$$

Proof: We first apply our uniform bound on the $M_4(v(t,\cdot))$ coming from Lemma 27 to conclude respectively that

$$\int_{-\infty}^{x} v\mathrm{d}t \ \leq \int_{-\infty}^{x}\left(\frac{t}{x}\right)^4 v(t)\mathrm{d}t \ \leq \frac{1}{x^4}\int_{-\infty}^{0} t^4 v(t)\mathrm{d}t \ \leq \min\left\{\frac{C_2}{x^4}, M\right\} \leq \frac{C_3}{1+x^4} \tag{310}$$

$$\int_x^\infty v\mathrm{d}t \ \le \int_x^\infty \left(\frac{t}{x}\right)^4 v(t)\mathrm{d}t \ \le \frac{1}{x^4}\int_0^\infty t^4 v(t)\mathrm{d}t \ \le \min\left\{\frac{C_2^*}{x^4}, M\right\} \le \frac{C_3^*}{1+x^4}. \tag{311}$$

Hence, by Lemma 27 and the Cauchy–Scwarz inequality,

$$\begin{aligned}
\left|\int_{-\infty}^0 \left(\int_{-\infty}^x v(v_{xxx}-t)\mathrm{d}t\right)\mathrm{d}x\right| &\le \int_{-\infty}^0 \left[\left(\int_{-\infty}^x v\mathrm{d}t\right)^{1/2}\left(\int_{-\infty}^x v(v_{xxx}-t)^2\mathrm{d}t\right)^{1/2}\right]\mathrm{d}x \\
&\le \int_{-\infty}^0 \left[\left(\int_{-\infty}^x \frac{C_3}{1+t^4}\mathrm{d}t\right)^{1/2}\left(\int_{-\infty}^0 v(v_{xxx}-t)^2\mathrm{d}t\right)^{1/2}\right]\mathrm{d}x \\
&\le \left[\int_{-\infty}^0 \left(\int_{-\infty}^x \frac{C_3}{1+t^4}\mathrm{d}t\right)^{1/2}\mathrm{d}x\right](D_E(v))^{1/2} \ .
\end{aligned} \tag{312}$$

$$\begin{aligned}
\left|\int_0^\infty \left(\int_x^\infty v(v_{xxx}-t)\mathrm{d}t\right)\mathrm{d}x\right| &\le \int_0^\infty \left[\left(\int_x^\infty v\mathrm{d}t\right)^{1/2}\left(\int_x^\infty v(v_{xxx}-t)^2\mathrm{d}t\right)^{1/2}\right]\mathrm{d}x \\
&\le \int_0^\infty \left[\left(\int_x^\infty \frac{C_3^*}{1+t^4}\mathrm{d}t\right)^{1/2}\left(\int_{-\infty}^0 v(v_{xxx}-t)^2\mathrm{d}t\right)^{1/2}\right]\mathrm{d}x \\
&\le \left[\int_0^\infty \left(\int_x^\infty \frac{C_3^*}{1+t^4}\mathrm{d}t\right)^{1/2}\mathrm{d}x\right](D_E(v))^{1/2} \ .
\end{aligned} \tag{313}$$

The remaining iterated integrals in (312) and (313) are clearly finite. The result then follows by the triangle inequality. □

4.4 Asymptotic equipartition

Lemma 30 : Under the same conditions imposed in Lemma 25, with the same constant K, we have that for all $T > 0$,

$$\inf_{T\le t\le 2T}\{\ |2\alpha(v(t,\cdot)) - 3\beta(v(t,\cdot))|\ \} \le \frac{KE^{1/2}(v_0)}{\sqrt{T}}\ . \tag{314}$$

Proof: For any $T > 0$, we have from Lemma 25 that

$$\frac{1}{T}\int_T^{2T} |2\alpha(v(t,\cdot)) - 3\beta(v(t,\cdot))|\mathrm{d}t \le \frac{1}{T}\int_T^{2T} KD_E^{1/2}(v(t,\cdot))\mathrm{d}t$$

By Cauchy–Schwarz inequality,

$$\begin{aligned}
\int_T^{2T} D_E^{1/2}(v(t,\cdot))\mathrm{d}t &\leq \sqrt{T}\left(\int_T^{2T} D_E(v(t,\cdot))\mathrm{d}t\right)^{1/2} \\
&\leq \sqrt{T}\left(\int_0^{\infty} D_E(v(t,\cdot))\mathrm{d}t\right)^{1/2} \\
&\leq \sqrt{T}\,(E(v_0))^{1/2}
\end{aligned} \tag{315}$$

Finally, $\inf_{T\leq t\leq 2T}\{\,|2\alpha(v(t,\cdot))-3\beta(v(t,\cdot))|\,\}$ is no greater than the average $|2\alpha(v(t,\cdot))-3\beta(v(t,\cdot))|$ over the interval $[T,2T]$. □

Lemma 31 : Under the same conditions imposed in Lemma 25, for all $t>0$,

$$|\beta(v(t,\cdot))-\beta(v^{(\infty)})| \leq Ce^{-t/2}\ . \tag{316}$$

Proof: As Carrillo and Toscani [27] have shown, there is a constant C so that

$$\|v(t,\cdot)-v^{(\infty)}\|_{L^1(\mathbb{R})} \leq CH(v(t,\cdot)|v^{(\infty)})\ ,$$

and thus there is another constant C so that

$$\|v(t,\cdot)-v^{(\infty)}\|_{L^1(\mathbb{R})} \leq Ce^{-t}\ ,$$

Now, for any $R>0$,

$$\begin{aligned}
|\beta(v(t,\cdot))-\beta(v^{(\infty)})| &\leq \int_{|x|<R} x^2|v(t,x)-v^{(\infty)}(x)|\mathrm{d}x + \int_{|x|>R} x^2|v(t,x)-v^{(\infty)}(x)|\mathrm{d}x \\
&\leq R^2\int_{|x|<R}|v(t,x)-v^{(\infty)}(x)|\mathrm{d}x + \frac{1}{R^2}\int_{|x|>R} x^4(|v(t,x)|+|v^{(\infty)}(x)|)\mathrm{d}x \\
&\leq c\left(R^2 s^{-t}+\frac{1}{R^2}\right)\ .
\end{aligned} \tag{317}$$

The optimal choice of R^2 is $R^2=e^{t/2}$, which yields the result. □

Proof of Theorem 24: Since

$$2(\alpha(v|v^{(\infty)})+\beta(v|v^{(\infty)})) = 5\beta(v|v^{(\infty)})+(2\alpha(v|v^{(\infty)})-3\beta(v|v^{(\infty)})) \leq 5\beta(v|v^{(\infty)})+|2\alpha(v|v^{(\infty)})-3\beta(v|v^{(\infty)})|\ ,$$

it follows from the last two lemmas and (296) that for some t in the interval $[T,2T]$,

$$
\begin{aligned}
2E(v|v^{(\infty)})) &\leq \{\, \alpha(v|v^{(\infty)}) + \beta(v|v^{(\infty)}) \,\} \\
&\leq 5Ce^{-t/2} + \frac{KE^{1/2}(v_0)}{\sqrt{T}}
\end{aligned}
\tag{318}
$$

Since $E(v|v^{(\infty)}))$ is monotone decreasing, this implies that

$$
E(v(2T,\cdot)|v^{(\infty)})) \leq Ce^{-T/2} + \frac{KE^{1/2}(v_0)}{\sqrt{T}} \; .
$$

Now, possessing this bound, we can go back and use it to improve (315). Doing so will give a bound in terms of $T^{-3/4}$. Returning to (315) again and using this yields a bound in terms of $T^{-7/8}$. Continuing, we can obtain a bound in terms of $T^{\epsilon-1}$ for any $\epsilon > 0$.

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